

# DUAL EQUIVALENCE GRAPHS I: A COMBINATORIAL PROOF OF LLT AND MACDONALD POSITIVITY

SAMI H. ASSAF

**ABSTRACT.** We make a systematic study of a new combinatorial construction called a dual equivalence graph. We axiomatize these graphs and prove that their generating functions are symmetric and Schur positive. By constructing a graph on ribbon tableaux which we transform into a dual equivalence graph, we give a combinatorial proof of the symmetry and Schur positivity of the ribbon tableaux generating functions introduced by Lascoux, Leclerc and Thibon. Using Haglund's formula for the transformed Macdonald polynomials, this also gives a combinatorial formula for the Schur expansion of Macdonald polynomials.

## 1. INTRODUCTION

The immediate purpose of this paper is to give a combinatorial formula for the Schur coefficients of LLT polynomials which, as a corollary, yields a combinatorial formula for the Schur coefficients of Macdonald polynomials. Our real purpose, however, is not only to obtain these results, but also to introduce a new combinatorial construction, called a *dual equivalence graph*, by which one can establish the symmetry and Schur positivity of functions expressed in terms of monomials.

The transformed Macdonald polynomials,  $\tilde{H}_\mu(x; q, t)$ , a transformation of the polynomials introduced by Macdonald [Mac88] in 1988, are defined to be the unique symmetric functions satisfying certain triangularity and orthogonality conditions. The existence of functions satisfying these conditions is a theorem, from which it follows that the  $\tilde{H}_\mu(x; q, t)$  form a basis for symmetric functions in two additional parameters. The *Kostka-Macdonald coefficients*, denoted  $\tilde{K}_{\lambda, \mu}(q, t)$ , give the change of basis from Macdonald polynomials to Schur functions, namely,

$$\tilde{H}_\mu(x; q, t) = \sum_{\lambda} \tilde{K}_{\lambda, \mu}(q, t) s_{\lambda}(x).$$

A priori,  $\tilde{K}_{\lambda, \mu}(q, t)$  is a rational function in  $q$  and  $t$  with rational coefficients, i.e.  $\tilde{K}_{\lambda, \mu}(q, t) \in \mathbb{Q}(q, t)$ .

The Macdonald Positivity Theorem [Hai01], first conjectured by Macdonald in 1988 [Mac88], states that  $\tilde{K}_{\lambda, \mu}(q, t)$  is in fact a polynomial in  $q$  and  $t$  with nonnegative integer coefficients, i.e.  $\tilde{K}_{\lambda, \mu}(q, t) \in \mathbb{N}[q, t]$ . Garsia and Haiman [GH93] conjectured that the transformed Macdonald polynomials  $\tilde{H}_\mu(x; q, t)$  could be realized as the bi-graded characters of certain modules for the diagonal action of the symmetric group  $S_n$  on two sets of variables. Once resolved, this conjecture gives a representation theoretic interpretation of Kostka-Macdonald coefficients as the graded multiplicity of an irreducible representation in the Garsia-Haiman module, and hence  $\tilde{K}_{\lambda, \mu}(q, t) \in \mathbb{N}[q, t]$ . Following an idea outlined by Procesi, Haiman [Hai01] proved this conjecture by analyzing the algebraic geometry of the isospectral Hilbert scheme of  $n$  points in the plane, consequently establishing Macdonald Positivity. This proof, however, is purely geometric and does not offer a combinatorial interpretation for  $\tilde{K}_{\lambda, \mu}(q, t)$ .

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The LLT polynomial  $\tilde{G}_\mu^{(k)}(x; q)$ , originally defined by Lascoux, Leclerc and Thibon [LLT97] in 1997, is the  $q$ -generating function of  $k$ -ribbon tableaux of shape  $\mu$  weighted by a statistic called *cospin*. By the Stanton-White correspondence [SW85],  $k$ -ribbon tableaux are in bijection with certain  $k$ -tuples of tableaux, from which it follows that LLT polynomials are  $q$ -analogs of products of Schur functions. More recently, an alternative definition of  $\tilde{G}_\mu^{(k)}(x; q)$  as the  $q$ -generating function of  $k$ -tuples of semi-standard tableaux of shapes  $\boldsymbol{\mu} = (\mu^{(0)}, \dots, \mu^{(k-1)})$  weighted by a statistic called  $k$ -inversions is given in [HHL<sup>+</sup>05b].

Using Fock space representations of quantum affine Lie algebras constructed by Kashiwara, Miwa and Stern [KMS95], Lascoux, Leclerc and Thibon [LLT97] proved that  $\tilde{G}_\mu^{(k)}(x; q)$  is a symmetric function. Thus we may define the Schur coefficients,  $\tilde{K}_{\lambda, \mu}^{(k)}(q)$ , by

$$\tilde{G}_\mu^{(k)}(x; q) = \sum_{\lambda} \tilde{K}_{\lambda, \mu}^{(k)}(q) s_{\lambda}(x).$$

Using Kazhdan-Lusztig theory, Leclerc and Thibon [LT00] proved that  $\tilde{K}_{\lambda, \mu}^{(k)}(q) \in \mathbb{N}[q]$  for straight shapes  $\mu$ . Grojnowski and Haiman [GH] recently extended this to skew shapes. Again, the proof of positivity is by a geometric argument, and as such offers no combinatorial description for  $\tilde{K}_{\lambda, \mu}^{(k)}(q)$ .

In 2004, Haglund [Hag04] conjectured a combinatorial formula for the monomial expansion of  $\tilde{H}_\mu(x; q, t)$ . Haglund, Haiman and Loehr [HHL05a] proved this formula using an elegant combinatorial argument, but this does not prove that  $\tilde{K}_{\lambda, \mu}(q, t) \in \mathbb{N}[q, t]$  since monomials are not Schur positive. Combining Theorem 2.3, Proposition 3.4 and equation (23) from [HHL05a], Haglund's formula expresses  $\tilde{H}_\mu(x; q, t)$  as a positive sum of LLT polynomials  $\tilde{G}_\nu^{(\mu_1)}(x; q)$  for certain skew shapes  $\nu$  depending on  $\mu$ . Therefore the LLT positivity result of Grojnowski and Haiman [GH] provides another proof of Macdonald positivity, though this proof is still non-combinatorial. One of the main purposes of this paper is to give a combinatorial proof of LLT positivity for arbitrary shapes, thereby completing the combinatorial proof of Macdonald positivity from Haglund's formula.

Combinatorial formulas for  $\tilde{K}_{\lambda, \mu}^{(k)}(q)$  and  $\tilde{K}_{\lambda, \mu}(q, t)$  have been found for certain special cases. In 1995, Carré and Leclerc [CL95] gave a combinatorial interpretation of  $\tilde{K}_{\lambda, \mu}^{(2)}(q)$  in their study of 2-ribbon tableaux, though a complete proof of their result wasn't found until 2005 by van Leeuwen [vL05] using the theory of crystal graphs. Also in 1995, Fishel [Fis95] gave the first combinatorial interpretation for  $\tilde{K}_{\lambda, \mu}(q, t)$  when  $\mu$  is a partition with 2 columns using rigged configurations. Other techniques have also led to formulas for the 2 column Macdonald polynomials [Zab99, LM03, Hag04], but in all cases, finding extensions for these formulas has proven elusive.

Following a suggestion from Haiman, we consider the dual equivalence relation on standard tableaux defined in [Hai92]. From this relation, Haiman suggested defining an edge-colored graph on standard tableaux and investigating how this graph may be related to the crystal graph on semi-standard tableaux. The result of this idea is a new combinatorial method for establishing the Schur positivity of a function expressed in terms of monomials. In this paper, this method is applied to LLT polynomials to obtain a combinatorial formula for  $\tilde{K}_{\lambda, \mu}^{(k)}(q)$ , and so, too, for  $\tilde{K}_{\lambda, \mu}(q, t)$ .

This paper is organized as follows. In Section 2, we review symmetric functions and the associated tableaux combinatorics. The theory of dual equivalence graphs is developed in Section 3, beginning in Section 3.1 with a review of dual equivalence and the construction of the graphs suggested by Haiman. In Section 3.2, we define a *dual equivalence graph* and present the structure theorem stating that every dual equivalence graph is isomorphic to one of the graphs from Section 3.1. On the symmetric function level, this shows that the generating function of a dual equivalence graph is symmetric and Schur positive and gives a combinatorial interpretation for the Schur coefficients. The proof of the theorem is left to Section 3.3.

The remainder of this paper contains the first application of this theory, beginning in Section 4 with the construction of a graph on  $k$ -tuples of tableaux. We present a reformulation of LLT polynomials in Section 4.1, and use it to describe the vertices and signatures of the graph. The

edges are constructed in Section 4.2 using a natural analog of dual equivalence. While these graphs are not, in general, dual equivalence graphs, we show in Section 5 that they can be transformed into dual equivalence graphs in a natural way that preserves the generating function. In particular, connected components of these graphs are Schur positive. The main consequence of this is a purely combinatorial proof of the symmetry and Schur positivity of LLT and Macdonald polynomials as well as a combinatorial formula for the Schur expansions.

Numerous examples of the graphs introduced in this paper are given in two appendices.

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## 2. PRELIMINARIES

**2.1. Partitions and tableaux.** We represent an integer *partition*  $\lambda$  by the decreasing sequence of its (nonzero) parts

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l), \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0.$$

We denote the size of  $\lambda$  by  $|\lambda| = \sum_i \lambda_i$  and the length of  $\lambda$  by  $l(\lambda) = \max\{i : \lambda_i > 0\}$ . If  $|\lambda| = n$ , we say that  $\lambda$  is a *partition of  $n$* . Let  $\geq$  denote the *dominance partial ordering* on partitions of  $n$ , defined by

$$(2.1) \quad \lambda \geq \mu \iff \lambda_1 + \lambda_2 + \dots + \lambda_i \geq \mu_1 + \mu_2 + \dots + \mu_i \quad \forall i.$$

A *composition*  $\pi$  is a finite sequence of non-negative integers  $\pi = (\pi_1, \pi_2, \dots, \pi_m), \pi_i \geq 0$ .

The *Young diagram* of a partition  $\lambda$  is the set of points  $(i, j)$  in the  $\mathbb{Z} \times \mathbb{Z}$  lattice such that  $1 \leq i \leq \lambda_j$ . We draw the diagram so that each point  $(i, j)$  is represented by the unit cell southwest of the point; see Figure 1. Abusing notation, we write  $\lambda$  for both the partition and its diagram.



FIGURE 1. The Young diagram for  $(5, 4, 4, 1)$  and the skew diagram for  $(5, 4, 4, 1)/(3, 2, 2)$ .

For partitions  $\lambda, \mu$ , we write  $\mu \subset \lambda$  whenever the diagram of  $\mu$  is contained within the diagram of  $\lambda$ ; equivalently  $\mu_i \leq \lambda_i$  for all  $i$ . In this case, we define the *skew diagram*  $\lambda/\mu$  to be the set theoretic difference  $\lambda - \mu$ , e.g. see Figure 1. For our purposes, we depart from the norm by *not* identifying skew shapes that are translates of one another. A *connected skew diagram* is one where exactly one cell has no cell immediately north or west of it, and exactly one cell has no cell immediately south or east of it. A *ribbon*, also called a *rim hook*, is a connected skew diagram containing no  $2 \times 2$  block.

A *filling* of a (skew) diagram  $\lambda$  is a map  $S : \lambda \rightarrow \mathbb{Z}_+$ . A *semi-standard Young tableau* is a filling which is weakly increasing along each row and strictly increasing along each column. A semi-standard Young tableau is *standard* if it is a bijection from  $\lambda$  to  $[n]$ , where  $[n] = \{1, 2, \dots, n\}$ . For  $\lambda$  a diagram of size  $n$ , define

$$\begin{aligned} \text{SSYT}(\lambda) &= \{\text{semi-standard tableaux } T : \lambda \rightarrow \mathbb{Z}_+\}, \\ \text{SYT}(\lambda) &= \{\text{standard tableaux } T : \lambda \rightarrow [n]\}. \end{aligned}$$

For  $T \in \text{SSYT}(\lambda)$ , we say that  $T$  has *shape*  $\lambda$ . If  $T$  contains entries  $1^{\pi_1}, 2^{\pi_2}, \dots$  for some composition  $\pi$ , then we say  $T$  has *weight*  $\pi$ .

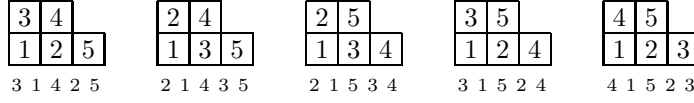


FIGURE 2. The standard Young tableaux of shape  $(3, 2)$  with their content reading words.

The *content* of a cell of a diagram indexes the diagonal on which it occurs, i.e.  $c(x) = i - j$  when the cell  $x$  lies in position  $(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ . The *content reading word* of a semi-standard tableaux is obtained by reading the entries in increasing order of content, going southwest to northeast along each diagonal (on which the content is constant). For examples, see Figure 2.

**2.2. Symmetric functions.** We have the familiar integral bases for  $\Lambda$ , the ring of symmetric functions, from [Mac95]: the monomial symmetric functions  $m_\lambda$ , the elementary symmetric functions  $e_\lambda$ , the complete homogeneous symmetric functions  $h_\lambda$ , and, most importantly, the *Schur functions*,  $s_\lambda$ , which may be defined in several ways. For the purposes of this paper, we take the tableau approach:

$$(2.2) \quad s_\lambda(x) = \sum_{T \in \text{SSYT}(\lambda)} x^T,$$

where  $x^T$  is the monomial  $x_1^{\pi_1} x_2^{\pi_2} \cdots$  when  $T$  has weight  $\pi$ . This formula also defines the *skew Schur functions*,  $s_{\lambda/\mu}$ , by taking the sum over semi-standard tableaux of shape  $\lambda/\mu$ .

The *Kostka numbers*,  $K_{\lambda, \mu}$ , give the change of basis from the complete homogeneous symmetric functions to the Schur functions and, dually, the change of basis from Schur functions to monomial symmetric functions, i.e.

$$h_\mu = \sum_{\lambda} K_{\lambda, \mu} s_\lambda; \quad s_\lambda = \sum_{\mu} K_{\lambda, \mu} m_\mu.$$

In particular,  $K_{\lambda, \mu}$  is the number of semi-standard Young tableaux of shape  $\lambda$  and weight  $\mu$ . For example,  $K_{(3, 2), (1^5)} = 5$  corresponding to the five standard Young tableaux of shape  $(3, 2)$  in Figure 2. Since the Schur functions are the characters of the irreducible representations of  $\text{GL}_n$ , the Kostka numbers also give weight multiplicities for  $\text{GL}_n$  modules. Throughout this paper, we are interested in certain one- and two-parameter generalizations of the Kostka numbers.

As we shall see in Section 3, it will often be useful to express a function in terms of Gessel's fundamental quasi-symmetric functions [Ges84] rather than monomials. For  $\sigma \in \{\pm 1\}^{n-1}$ , the *fundamental quasi-symmetric function*  $Q_\sigma(x)$  is defined by

$$(2.3) \quad Q_\sigma(x) = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ i_j = i_{j+1} \Rightarrow \sigma_j = +1}} x_{i_1} \cdots x_{i_n}.$$

We have indexed quasi-symmetric functions by sequences of  $+1$ 's and  $-1$ 's, though by setting  $D(\sigma) = \{i | \sigma_i = -1\}$ , we may change the indexing to subsets of  $[n-1]$ . Similarly, letting  $\pi(\sigma)$  be the composition defined by setting  $\pi_1 + \cdots + \pi_i$  to be the position of the  $i$ th  $-1$ , where here we regard  $\sigma_n = -1$  as the final  $-1$ , we may change the indexing to compositions of  $n$ .

To connect quasi-symmetric functions with Schur functions, for  $T$  a standard tableau on  $[n]$  with content reading word  $w_T$ , define the *descent signature*  $\sigma(T) \in \{\pm 1\}^{n-1}$  by

$$(2.4) \quad \sigma(T)_i = \begin{cases} +1 & \text{if } i \text{ appears to the left of } i+1 \text{ in } w_T \\ -1 & \text{if } i+1 \text{ appears to the left of } i \text{ in } w_T \end{cases}.$$

For example, the descent signatures for the tableaux in Figure 2 are  $+ - + +$ ,  $- + - +$ ,  $- + + -$ ,  $+ - + -$ ,  $+ + - +$ , from left to right. Note that if we replace the content reading word with either the row or column reading word, the resulting sequence in (2.4) remains unchanged.

**Proposition 2.1** ([Ges84]). *The Schur function  $s_\lambda$  is expressed in terms of quasi-symmetric functions by*

$$(2.5) \quad s_\lambda(x) = \sum_{T \in \text{SYT}(\lambda)} Q_{\sigma(T)}(x).$$

Comparing (2.2) with (2.5), using quasi-symmetric functions instead of monomials allows us to work with standard tableaux rather than semi-standard tableaux. One advantage of this formula is that unlike (2.2), the right hand side of (2.5) is finite. Continuing with the example in Figure 2,

$$s_{(3,2)}(x) = Q_{+---}(x) + Q_{-++-}(x) + Q_{-+-+}(x) + Q_{+-+-}(x) + Q_{++--}(x).$$

**2.3. LLT polynomials.** Lascoux, Leclerc and Thibon [LLT97] originally defined  $\tilde{G}_\mu^{(k)}(x; q)$  to be the  $q$ -generating function of  $k$ -ribbon tableaux of shape  $\mu$  weighted by cospin. Below we give an alternative definition of  $\tilde{G}_\mu^{(k)}(x; q)$  as the  $q$ -generating function of  $k$ -tuples of semi-standard tableaux of shapes  $\boldsymbol{\mu} = (\mu^{(0)}, \dots, \mu^{(k-1)})$  weighted by  $k$ -inversions first presented in [HHL<sup>+</sup>05b]. For a detailed account of the equivalence of these definitions (actually  $q^a \tilde{G}_\mu^{(k)}(x; q) = \tilde{G}_\mu^{(k)}(x; q)$  for a constant  $a \geq 0$  depending on  $\mu$ ), see [HHL<sup>+</sup>05b, Ass07].

Extending prior notation, define

$$\begin{aligned} \text{SSYT}_k(\boldsymbol{\lambda}) &= \{\text{semi-standard } k\text{-tuples of tableaux of shapes } (\lambda^{(0)}, \dots, \lambda^{(k-1)})\}, \\ \text{SYT}_k(\boldsymbol{\lambda}) &= \{\text{standard } k\text{-tuples of tableaux of shapes } (\lambda^{(0)}, \dots, \lambda^{(k-1)})\}. \end{aligned}$$

As with tableaux, if  $\mathbf{T} = (T^{(0)}, \dots, T^{(k-1)}) \in \text{SSYT}_k(\boldsymbol{\lambda})$  has entries  $1^{\pi_1}, 2^{\pi_2}, \dots$ , then we say that  $\mathbf{T}$  has *shape*  $\boldsymbol{\lambda}$  and *weight*  $\pi$ . Note that a standard  $k$ -tuple of tableaux has weight  $(1^n)$ , e.g. see Figure 3, and this is not the same as a  $k$ -tuple of standard tableaux, which has weight  $(1^{m_1}, 2^{m_2}, \dots)$  where  $m_i$  is the number of shapes of size at least  $i$ .

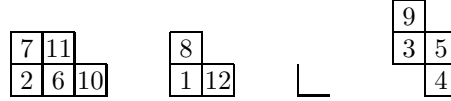


FIGURE 3. A standard 4-tuple of shape  $((3,2), (2,1), \emptyset, (2,2,1)/(1))$

For a  $k$ -tuple of (skew) shapes  $(\lambda^{(0)}, \dots, \lambda^{(k-1)})$ , define the *shifted content* of a cell  $x$  by

$$(2.6) \quad \tilde{c}(x) = k \cdot c(x) + i$$

when  $x$  is a cell of  $\lambda^{(i)}$ , where  $c(x)$  is the usual content of  $x$  regarded as a cell of  $\lambda^{(i)}$ . For  $\mathbf{T} \in \text{SSYT}_k$ , let  $\mathbf{T}(x)$  denote the entry of the cell  $x$  in  $\mathbf{T}$ . Define the *set of  $k$ -inversions of  $\mathbf{T}$*  by

$$(2.7) \quad \text{Inv}_k(\mathbf{T}) = \{(x, y) \mid k > \tilde{c}(y) - \tilde{c}(x) > 0 \text{ and } \mathbf{T}(x) > \mathbf{T}(y)\}.$$

Then the  *$k$ -inversion number of  $\mathbf{T}$*  is given by

$$(2.8) \quad \text{inv}_k(\mathbf{T}) = |\text{Inv}_k(\mathbf{T})|.$$

For example, suppose  $\mathbf{T}$  is the 4-tuple of tableaux in Figure 3. Since  $\mathbf{T}$  is standard, let us abuse notation by representing a cell of  $\mathbf{T}$  by the entry it contains. Then the set of 4-inversions is

$$\text{Inv}_4(\mathbf{T}) = \left\{ \begin{array}{l} (9, 7), (9, 8), (7, 3), (8, 3), (8, 2), (3, 2), (3, 1), \\ (2, 1), (11, 1), (11, 5), (6, 4), (12, 4), (12, 10) \end{array} \right\},$$

and so  $\text{inv}_4(\mathbf{T}) = 13$ .

By [HHL<sup>+</sup>05b], the LLT polynomial  $\tilde{G}_\mu^{(k)}(x; q)$  is given by

$$(2.9) \quad \tilde{G}_\mu^{(k)}(x; q) = \sum_{\mathbf{T} \in \text{SSYT}_k(\boldsymbol{\mu})} q^{\text{inv}_k(\mathbf{T})} x^{\mathbf{T}},$$

where  $x^{\mathbf{T}}$  is the monomial  $x_1^{\pi_1} x_2^{\pi_2} \cdots$  when  $\mathbf{T}$  has weight  $\pi$ .

Notice that when  $q = 1$ , (2.9) reduces to a product of Schur functions:

$$(2.10) \quad \sum_{\mathbf{T} \in \text{SSYT}_k(\lambda)} x^{\mathbf{T}} = \prod_{i=0}^{k-1} \sum_{T^{(i)} \in \text{SSYT}(\lambda^{(i)})} x^{T^{(i)}} = \prod_{i=0}^{k-1} s_{\lambda^{(i)}}(x).$$

Define the *content reading word* of a  $k$ -tuple of tableaux to be the word obtained by reading entries in increasing order of shifted content and reading diagonals southwest to northeast. For the example in Figure 3, the content reading word is (9, 7, 8, 3, 2, 11, 1, 5, 6, 12, 4, 10).

For  $\mathbf{T}$  a standard  $k$ -tuple of tableaux, define  $\sigma(\mathbf{T})$  analogously to (2.4) using the content reading word. Expressed in terms of quasi-symmetric functions, (2.9) becomes

$$(2.11) \quad \tilde{G}_{\mu}^{(k)}(x; q) = \sum_{\mathbf{T} \in \text{SYT}_k(\mu)} q^{\text{inv}_k(\mathbf{T})} Q_{\sigma(\mathbf{T})}(x).$$

One of the main goals of this paper is to understand the Schur coefficients of  $\tilde{G}_{\mu}^{(k)}(x; q)$  defined by

$$\tilde{G}_{\mu}^{(k)}(x; q) = \sum_{\lambda} \tilde{K}_{\lambda, \mu}^{(k)}(q) s_{\lambda}(x).$$

In particular, we will show that  $\tilde{K}_{\lambda, \mu}^{(k)}(q)$  is a polynomial in  $q$  with nonnegative integer coefficients.

**2.4. Macdonald polynomials.** The transformed Macdonald polynomials  $\tilde{H}_{\mu}(x; q, t)$  were originally defined by Macdonald [Mac88] to be the unique symmetric functions satisfying certain orthogonality and triangularity conditions. Haglund's monomial expansion for Macdonald polynomials [Hag04, HHL05a] gives an alternative combinatorial definition of  $\tilde{H}_{\mu}(x; q, t)$  as the  $q, t$ -generating functions for fillings of the diagram of  $\mu$ , e.g. see Figure 4. Since the proof of the equivalence of these two definitions is purely combinatorial [HHL05a], we will use the latter characterization.

For a cell  $x$  in the diagram of  $\lambda$ , define the *arm* of  $x$  to be the set of cells east of  $x$ , and the *leg* of  $x$  to be the set of cells north of  $x$ . Denote the sizes of the arm and leg of  $x$  by  $a(x)$  and  $l(x)$ , respectively. For example, letting  $x$  denote the cell with entry 3 in the filling in Figure 4, the arm of  $x$  consists of the cells with entries 4 and 10 and the leg of  $x$  consists of the cell with entry 14, and so we have  $a(x) = 2$  and  $l(x) = 1$ .

5				
11	14	9	2	
6	3	4	10	
8	1	13	7	12

FIGURE 4. A standard filling of shape  $(5, 4, 4, 1)$ .

Let  $S$  be a filling of a partition  $\lambda$ . A *descent* of  $S$  is a cell  $c$  of  $\lambda$ , not in the first row, such that the entry in  $c$  is greater than the entry in the cell immediately south of  $c$ . Denote by  $\text{Des}(S)$  the set of all descents of  $S$ , i.e.

$$(2.12) \quad \text{Des}(S) = \{(i, j) \in \lambda \mid j > 1 \text{ and } S(i, j) > S(i, j-1)\}.$$

Define the *major index* of  $S$ , denoted  $\text{maj}(S)$ , by

$$(2.13) \quad \text{maj}(S) \stackrel{\text{def}}{=} |\text{Des}(S)| + \sum_{c \in \text{Des}(S)} l(c).$$

Note that for  $\mu = (1^n)$ , this gives the usual major index for the reading word of the filling.

For example, let  $S$  be the filling in Figure 4. As before, let us abuse notation by representing a cell of  $S$  by the entry which it contains. Then the descents of  $S$  are  $\text{Des}(S) = \{11, 14, 9, 3, 10\}$ , and so the major index of  $S$  is  $\text{maj}(S) = 5 + (1 + 0 + 0 + 1 + 1) = 8$ .

An ordered pair of cells  $(c, d)$  is called *attacking* if  $c$  and  $d$  lie in the same row with  $c$  to the west of  $d$ , or if  $c$  is in the row immediately north of  $d$  and  $c$  lies strictly east of  $d$ . An *inversion pair* of  $S$  is an attacking pair  $(c, d)$  such that the entry in  $c$  is greater than the entry in  $d$ . Denote by  $\text{Inv}(S)$  the set of inversion pairs of  $S$ , i.e.

$$(2.14) \quad \text{Inv}(S) = \left\{ ((i, j), (g, h)) \in \lambda \mid \begin{array}{l} j = h \text{ and } i < g \text{ or } j = h + 1 \text{ and } g < i, \\ \text{and } S(i, j) > S(g, h) \end{array} \right\}.$$

Define the *inversion number* of  $S$ , denoted  $\text{inv}(S)$ , by

$$(2.15) \quad \text{inv}(S) \stackrel{\text{def}}{=} |\text{Inv}(S)| - \sum_{c \in \text{Des}(S)} a(c).$$

Note that for  $\mu = (n)$ , this gives the usual inversion number for the reading word of the filling.

For our running example, the inversion pairs of  $S$  are given by

$$\text{Inv}(S) = \left\{ \begin{array}{llllll} (11, 9), & (14, 2), & (9, 6), & (6, 4), & (10, 1), & (13, 7), \\ (11, 2), & (14, 6), & (9, 3), & (4, 1), & (8, 1), & (13, 12) \\ (14, 9), & (9, 2), & (6, 3), & (10, 8), & (8, 7), & \end{array} \right\},$$

and so the inversion number of  $S$  is  $\text{inv}(S) = 17 - (3 + 2 + 1 + 2 + 0) = 9$ .

*Remark 2.2.* If  $c \in \text{Des}(S)$ , say with  $d$  the cell of  $S$  immediately south of  $c$ , then for every cell  $e$  of the arm of  $c$ , the entry in  $e$  is either bigger than the entry in  $d$  or smaller than the entry in  $c$  (or both). In the former case,  $(e, d)$  will form an inversion pair, and in the latter case,  $(c, e)$  will form an inversion pair. Thus every triple of cells  $(c, e, d)$  with  $d$  immediately south of  $c$  and  $e$  in the arm of  $c$  contributes at least one inversion to  $\text{inv}(S)$ , and so  $\text{inv}(S)$  is a non-negative integer.

By [HHL05a], the transformed Macdonald polynomial  $\tilde{H}_\mu(x; q, t)$  is given by

$$(2.16) \quad \tilde{H}_\mu(x; q, t) = \sum_{S: \mu \rightarrow \mathbb{Z}_+} q^{\text{inv}(S)} t^{\text{maj}(S)} x^S = \sum_{S: \mu \rightsquigarrow [n]} q^{\text{inv}(S)} t^{\text{maj}(S)} Q_{\sigma(S)},$$

where  $\sigma(S)$  is defined analogously to (2.4) using the *row reading word* of a standard filling  $S$ . For example, the row reading word for the standard filling in Figure 4 is  $(5, 11, 14, 9, 2, 6, 3, 4, 10, 8, 1, 13, 7, 12)$ . Again, our main objective is to understand the Schur coefficients defined by

$$(2.17) \quad \tilde{H}_\mu(x; q, t) = \sum_{\lambda} \tilde{K}_{\lambda, \mu}(q, t) s_{\lambda}(x).$$

In this paper, we give a combinatorial proof that  $\tilde{K}_{\lambda, \mu}(q, t)$  is a polynomial in  $q$  and  $t$  with nonnegative integer coefficients. This proof is a corollary to the proof for  $\tilde{K}_{\lambda, \mu}^{(k)}(q)$  as we now explain.

The expression in (2.16) is related to LLT polynomials as follows. Let  $D$  be a possible descent set for  $\mu$ , i.e.  $D$  is a collection cells of  $\mu/(\mu_1)$ . For  $i = 1, \dots, \mu_1$ , let  $\mu_D^{(i-1)}$  be the ribbon obtained from the  $i$ th column of  $\mu$  by putting the cell  $(i, j)$  immediately south of  $(i, j + 1)$  if  $(i, j + 1) \in D$  and immediately east of  $(i, j + 1)$  otherwise. Translate each  $\mu_D^{(i)}$  so that the southeastern most cell has shifted content  $n + i$  for some (any) fixed integer  $n$ . Then each filling  $S$  of shape  $\mu$  with  $\text{Des}(S) = D$  may be regarded as a semi-standard  $\mu_1$ -tuple of tableaux of shape  $\mu_D$ , denoted  $\mathbf{S}$ . For example, the filling  $S$  of shape  $(5, 4, 4, 1)$  in Figure 4 corresponds to the 5-tuple of ribbons of shapes  $(3, 3, 3, 2)/(3, 3, 1)$ ,  $(1, 1, 1)$ ,  $(2, 2, 1)/(2)$ ,  $(2, 2, 1)/(2, 1)$ ,  $(1)$ ; see Figure 5.

For this correspondence, it is crucial that we do not identify skew shapes that are translates of one another. For example, the row reading word of the filling in Figure 4 is precisely the content reading word of 5-tuple in Figure 5, but this is not the case if the first tableau is instead considered to have shape  $(3, 2)/(1)$ . Furthermore, the inversion pairs of  $S$  as defined in (2.14) correspond precisely with the  $\mu_1$ -inversions of  $\mathbf{S}$  as defined in (2.7). Since the major index statistic depends only on the descent set, for a given descent set  $D$  we may define  $\text{maj}(D)$  by  $\text{maj}(D) = \text{maj}(S)$  for any filling

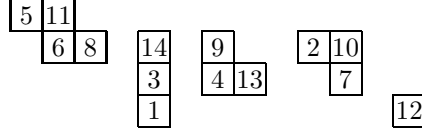


FIGURE 5. A standard filling of shape  $(5, 4, 4, 1)$  transformed into a 5-tuple of ribbons of shapes  $(3, 3, 3, 2)/(3, 3, 1)$ ,  $(1, 1, 1)$ ,  $(2, 2, 1)/(2)$ ,  $(2, 2, 2)/(2, 1)$ ,  $(1)$ .

$S$  with  $\text{Des}(S) = D$ . Similarly, define  $a(D) = \sum_{c \in D} a(c)$ . Then we may rewrite (2.16) in terms of LLT polynomials as

$$(2.18) \quad \tilde{H}_\mu(x; q, t) = \sum_{D \subseteq \mu/(\mu_1)} q^{-a(D)} t^{\text{maj}(D)} \tilde{G}_{\mu_D}^{(\mu_1)}(x; q).$$

Note that each term of  $\tilde{G}_{\mu_D}^{(\mu_1)}(x; q)$  contains a factor of  $q^a$  for some  $a \geq a(D)$  (in fact, this is the same constant mentioned in Section 2.3). In terms of Schur expansions, (2.18) may also be expressed as

$$(2.19) \quad \tilde{K}_{\lambda, \mu}(q, t) = \sum_{D \subseteq \mu/(\mu_1)} q^{-a(D)} t^{\text{maj}(D)} \tilde{K}_{\lambda, \mu_D}^{(\mu_1)}(q).$$

By the previous remark, proving  $\tilde{K}_{\lambda, \mu_D}^{(\mu_1)}(q) \in \mathbb{N}[q]$  consequently proves  $\tilde{K}_{\lambda, \mu}(q, t) \in \mathbb{N}[q, t]$ .

### 3. DUAL EQUIVALENCE GRAPHS

**3.1. The standard dual equivalence graph.** Dual equivalence was first defined by Haiman [Hai92] as a relation on tableaux dual to *jeu de taquin* equivalence under the Schensted correspondence. We use this relation to construct a graph whose vertices are standard tableaux and edges are elementary dual equivalence relations. Using quasi-symmetric functions, we define the generating function on the vertices of these graphs, thereby providing the connection with symmetric functions.

We begin by recalling the definition of dual equivalence on permutations regarded as words on  $[n]$ , which we extend to standard Young tableaux via the content reading word.

**Definition 3.1** ([Hai92]). An *elementary dual equivalence* on three consecutive letters  $i-1, i, i+1$  of a permutation is given by switching the outer two letters whenever the middle letter is not  $i$ :

$$\cdots i \cdots i \pm 1 \cdots i \mp 1 \cdots \equiv^* \cdots i \mp 1 \cdots i \pm 1 \cdots i \cdots$$

Two permutations are *dual equivalent* if they differ by some sequence of elementary dual equivalences. Two standard tableaux of the same shape are *dual equivalent* if their content reading words are.

Construct an edge-colored graph on standard tableaux of partition shape from the dual equivalence relation in the following way. Whenever two standard tableaux  $T, U$  have content reading words that differ by an elementary dual equivalence for  $i-1, i, i+1$ , connect  $T$  and  $U$  with an edge colored by  $i$ . Recall the definition of the content reading word  $w_T$  and the *descent signature* of a standard tableau  $T$  from (2.4):

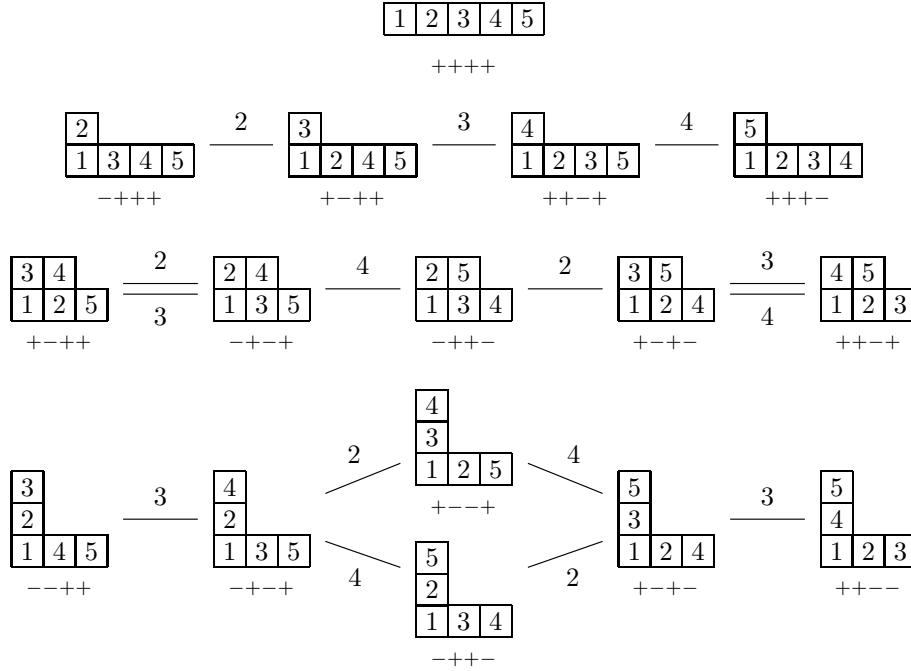
$$\sigma(T)_i = \begin{cases} +1 & \text{if } i \text{ appears to the left of } i+1 \text{ in } w_T \\ -1 & \text{if } i+1 \text{ appears to the left of } i \text{ in } w_T \end{cases}.$$

We associate to each tableau  $T$  the signature  $\sigma(T)$ . Several examples are given in Figure 6, and several more in Appendix A.

The connected components of the graph so constructed are the dual equivalence classes of standard tableaux. Let  $\mathcal{G}_\lambda$  denote the subgraph on tableaux of shape  $\lambda$ . The following proposition tells us that the  $\mathcal{G}_\lambda$  exactly give the connected components of the graph.

**Proposition 3.2** ([Hai92]). *Two standard tableaux on partition shapes are dual equivalent if and only if they have the same shape.*



FIGURE 6. The standard dual equivalence graphs  $\mathcal{G}_5, \mathcal{G}_{4,1}, \mathcal{G}_{3,2}$  and  $\mathcal{G}_{3,1,1}$ .

Define the generating function associated to  $\mathcal{G}_\lambda$  by

$$(3.1) \quad \sum_{v \in V(\mathcal{G}_\lambda)} Q_{\sigma(v)}(x) = s_\lambda(x).$$

By Proposition 2.1, this is Gessel's quasi-symmetric function expansion for a Schur function. In particular, the generating function of any vertex-signed graph whose connected components are isomorphic to the graphs  $\mathcal{G}_\lambda$  is automatically Schur positive. This observation is the main idea behind the following method for establishing the symmetry and Schur positivity of a function expressed in terms of fundamental quasi-symmetric functions. We will realize the given function as the generating function for a vertex-signed, edge-colored graph such that connected components of the graph are isomorphic to the graphs  $\mathcal{G}_\lambda$ . Therefore the connected components of the graph will correspond precisely to terms in the Schur expansion of the given function.

**3.2. Axiomatization of dual equivalence.** In this section, we characterize  $\mathcal{G}_\lambda$  in terms of edges and signatures so that we can readily identify those graphs that are isomorphic to some  $\mathcal{G}_\lambda$ .

**Definition 3.3.** A *signed, colored graph of type  $(n, N)$*  consists of the following data:

- a finite vertex set  $V$ ;
- a signature function  $\sigma : V \rightarrow \{\pm 1\}^{N-1}$ ;
- for each  $1 < i < n$ , a collection  $E_i$  of pairs of distinct vertices of  $V$ .

We denote such a graph by  $\mathcal{G} = (V, \sigma, E_2 \cup \dots \cup E_{n-1})$  or simply  $(V, \sigma, E)$ .

**Definition 3.4.** A signed, colored graph  $\mathcal{G} = (V, \sigma, E)$  of type  $(n, N)$  is a *dual equivalence graph of type  $(n, N)$*  if  $n \leq N$  and the following hold:

- (ax1) For  $w \in V$  and  $1 < i < n$ ,  $\sigma(w)_{i-1} = -\sigma(w)_i$  if and only if there exists  $x \in V$  such that  $\{w, x\} \in E_i$ . Moreover,  $x$  is unique when it exists.

- (ax2) For  $\{w, x\} \in E_i$ ,  $\sigma(w)_j = -\sigma(x)_j$  for  $j = i-1, i$ , and  $\sigma(w)_h = \sigma(x)_h$  for  $h < i-2$  and  $h > i+1$ .
- (ax3) For  $\{w, x\} \in E_i$ , if  $\sigma(w)_{i-2} = -\sigma(x)_{i-2}$ , then  $\sigma(w)_{i-2} = -\sigma(w)_{i-1}$ , and if  $\sigma(w)_{i+1} = -\sigma(x)_{i+1}$ , then  $\sigma(w)_{i+1} = -\sigma(w)_i$ .
- (ax4) Every connected component of  $(V, \sigma, E_{i-1} \cup E_i)$  appears in Figure 7 and every connected component of  $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$  appears in Figure 8.
- (ax5) If  $\{w, x\} \in E_i$  and  $\{x, y\} \in E_j$  for  $|i - j| \geq 3$ , then  $\{w, v\} \in E_j$  and  $\{v, y\} \in E_i$  for some  $v \in V$ .
- (ax6) Any two vertices of a connected component of  $(V, \sigma, E_2 \cup \dots \cup E_i)$  may be connected by a path crossing at most one  $E_i$  edge.

Note that if  $n > 4$ , then the allowed structure for connected components of  $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$  dictates that every connected component of  $(V, \sigma, E_{i-1} \cup E_i)$  appears in Figure 7.

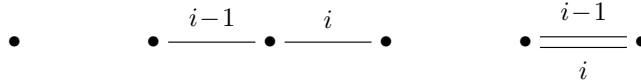


FIGURE 7. Allowed 2-color connected components of a dual equivalence graph.

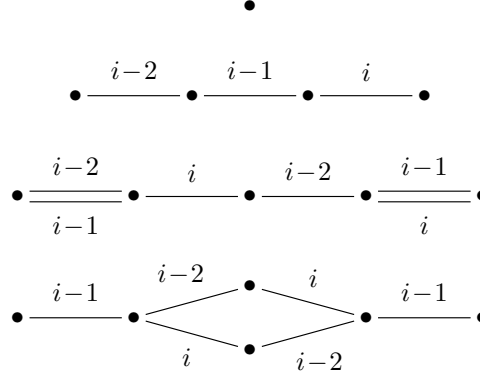


FIGURE 8. Allowed 3-color connected components of a dual equivalence graph.

Every connected component of a dual equivalence graph of type  $(n, N)$  is again a dual equivalence graph of type  $(n, N)$ .

It is often useful to consider a restricted set of edges of a signed, colored graph. To be precise, for  $m \leq n$  and  $M \leq N$ , the  $(m, M)$ -restriction of a signed, colored graph  $\mathcal{G}$  of type  $(n, N)$  consists of the vertex set  $V$ , signature function  $\sigma : V \rightarrow \{\pm 1\}^{M-1}$  obtained by truncating  $\sigma$  at  $M-1$ , and the edge set  $E_2 \cup \dots \cup E_{m-1}$ . For  $m \leq n, M \leq N$ , the  $(m, M)$ -restriction of a dual equivalence graph of type  $(n, N)$  is a dual equivalence graph of type  $(m, M)$ .

The graph for  $\mathcal{G}_{\lambda'}$  is obtained from  $\mathcal{G}_{\lambda}$  by conjugating each standard tableau and multiplying the signatures coordinate-wise by  $-1$ . Therefore the structure of  $\mathcal{G}_{(2,1,1,1)}, \mathcal{G}_{(2,2,1)}$  and  $\mathcal{G}_{(1,1,1,1,1)}$  is also indicated by Figure 6. Comparing this with Figure 8, axiom 4 stipulates that the restricted components of a dual equivalence graph are exactly the graphs for  $\mathcal{G}_{\lambda}$  when  $\lambda$  is a partition of 5.

**Proposition 3.5.** *For  $\lambda$  a partition of  $n$ ,  $\mathcal{G}_{\lambda}$  is a dual equivalence graph of type  $(n, n)$ .*

*Proof.* For  $T \in \text{SYT}(\lambda)$ ,  $\sigma(T)_{i-1} = -\sigma(T)_i$  if and only if  $i$  does not lie between  $i-1$  and  $i+1$  in the content reading word of  $T$ . In this case, there exists  $U \in \text{SYT}(\lambda)$  such that  $T$  and  $U$  differ by an elementary dual equivalence for  $i-1, i, i+1$ . Therefore  $U$  is obtained from  $T$  by swapping  $i$  with  $i-1$

or  $i+1$ , whichever lies further away, with the result that  $\sigma(T)_j = -\sigma(U)_j$  for  $j = i-1, i$  and also  $\sigma(T)_h = \sigma(U)_h$  for  $h < i-2$  and  $i+1 < h$ . This verifies axioms 1 and 2.

For axiom 3, if  $\sigma(T)_{i-2} = -\sigma(U)_{i-2}$ , then  $i$  and  $i-1$  have interchanged positions with  $i-2$  lying between, so that  $T$  and  $U$  also differ by an elementary dual equivalence for  $i-2, i-1, i$ , and similarly for  $i+1$ . From this, we obtain an explicit description of double edges, and so axiom 4 becomes a straightforward, finite check. If  $|i-j| \geq 3$ , then  $\{i-1, i, i+1\} \cap \{j-1, j, j+1\} = \emptyset$ , so the elementary dual equivalences for  $i-1, i, i+1$  and for  $j-1, j, j+1$  commute, thereby demonstrating axiom 5.

Finally, for  $T, U \in \text{SYT}(\lambda)$ ,  $|\lambda| = i+1$ , we must show that there exists a path from  $T$  to  $U$  crossing at most one  $E_i$  edge. Let  $\mathcal{C}_T$  (resp.  $\mathcal{C}_U$ ) denote the connected component of the  $(i, i)$ -restriction of  $\mathcal{G}_\lambda$  containing  $T$  (resp.  $U$ ). Let  $\mu$  (resp.  $\nu$ ) be the shape of  $T$  (resp.  $U$ ) with the cell containing  $i+1$  removed. Then  $\mathcal{C}_T \cong \mathcal{G}_\mu$  and  $\mathcal{C}_U \cong \mathcal{G}_\nu$ . If  $\mu = \nu$ , then, by Proposition 3.2,  $\mathcal{C}_T = \mathcal{C}_U$  and axiom 6 holds. Assume, then, that  $\mu \neq \nu$ . Since  $\mu, \nu \subset \lambda$  and  $|\mu| = |\nu| = |\lambda| - 1$ , both cells  $\lambda/\mu$  and  $\lambda/\nu$  must be northeastern corners of  $\lambda$ . Therefore there exists  $T' \in \text{SYT}(\lambda)$  with  $i$  in position  $\lambda/\nu$ ,  $i+1$  in position  $\lambda/\mu$ , and  $i-1$  between  $i$  and  $i+1$  in the content reading word of  $T'$ . Let  $U'$  be the result of swapping  $i$  and  $i+1$  in  $T'$ , in particular,  $\{T', U'\} \in E_i$ . By Proposition 3.2,  $T'$  is in  $\mathcal{C}_T$  and  $U'$  is in  $\mathcal{C}_U$ , hence there exists a path from  $T$  to  $T'$  and a path from  $U'$  to  $U$  each crossing only edges  $E_h$ ,  $h < i$ . This establishes axiom 6.  $\square$

*Remark 3.6.* For partitions  $\lambda \subset \rho$ , with  $|\lambda| = n$  and  $|\rho| = N$ , choose a tableau  $A$  of shape  $\rho/\lambda$  with entries  $n+1, \dots, N$ . Define the set of standard Young tableaux of shape  $\lambda$  augmented by  $A$ , denoted  $\text{ASYT}(\lambda, A)$ , to be those  $T \in \text{SYT}(\rho)$  such that  $T$  restricted to  $\rho/\lambda$  is  $A$ . Let  $\mathcal{G}_{\lambda, A}$  be the signed, colored graph of type  $(n, N)$  constructed on  $\text{ASYT}(\lambda, A)$  with  $i$ -edges given by elementary dual equivalences for  $i-1, i, i+1$  with  $i < n$ . Then  $\mathcal{G}_{\lambda, A}$  is a dual equivalence graph of type  $(n, N)$ , and the  $(n, n)$ -restriction of  $\mathcal{G}_{\lambda, A}$  is  $\mathcal{G}_\lambda$ .

Proposition 3.5 is the first step towards justifying Definition 3.4, and also allows us to refer to  $\mathcal{G}_\lambda$  as the *standard dual equivalence graph corresponding to  $\lambda$* . In order to show the converse, we first need the notion of a morphism between two signed, colored graphs.

**Definition 3.7.** A *morphism* between two signed, colored graphs of type  $(n, N)$ , say  $\mathcal{G} = (V, \sigma, E)$  and  $\mathcal{H} = (W, \tau, F)$ , is a map  $\phi : V \rightarrow W$  such that for every  $u, v \in V$

- for every  $1 \leq i < N$ , we have  $\sigma(v)_i = \tau(\phi(v))_i$ , and
- for every  $1 < i < n$ , if  $\{u, v\} \in E_i$ , then  $\{\phi(u), \phi(v)\} \in F_i$ .

A morphism is an *isomorphism* if it is a bijection on vertex sets.

When two graphs satisfy axiom 1, as all graphs in this paper do, an isomorphism between them is a sign-preserving bijection on vertex sets that respects color-adjacency.

*Remark 3.8.* If  $\phi$  is a morphism from a signed, colored graph  $\mathcal{G}$  of type  $(n, N)$  satisfying axiom 1 to an augmented standard dual equivalence graph  $\mathcal{G}_{\lambda, A}$ , then  $\phi$  is surjective. Indeed, suppose  $T = \phi(v)$  for some  $T \in \text{ASYT}(\lambda, A)$  and some vertex  $v$  of  $\mathcal{G}$ . Then for every  $1 < i < n$ , if  $\{T, U\} \in E_i$ , then since  $\sigma(v) = \sigma(T)$ , by axiom 1 there exists a unique vertex  $w$  of  $\mathcal{G}$  such that  $\{v, w\} \in E_i$  in  $\mathcal{G}$ . Since  $\phi$  is a morphism, we must have  $\{T, \phi(w)\} \in E_i$  in  $\mathcal{G}_{\lambda, A}$ . Thus by the uniqueness condition of axiom 1,  $\phi(w) = U$ , and so  $U$  also lies in the image of  $\phi$ . Therefore the  $i$ -neighbor of any vertex in the image of  $\phi$  also lies in the image since  $\phi$  preserves  $i$ -edges. Since  $\mathcal{G}_{\lambda, A}$  is connected,  $\phi$  is surjective.

The final justification of this axiomatization is the following converse of Proposition 3.5.

**Theorem 3.9.** *Every connected component of a dual equivalence graph of type  $(n, n)$  is isomorphic to  $\mathcal{G}_\lambda$  for a unique partition  $\lambda$  of  $n$ .*

The proof of Theorem 3.9 is postponed until Section 3.3. We conclude this section by interpreting Theorem 3.9 in terms of symmetric functions.

**Corollary 3.10.** *Let  $\mathcal{G}$  be a dual equivalence graph of type  $(n, n)$  such that every vertex is assigned some additional statistic  $\alpha$ . Let  $C(\lambda)$  denote the set of connected components of  $\mathcal{G}$  that are isomorphic to  $\mathcal{G}_\lambda$ . If  $\alpha$  is constant on connected components of  $\mathcal{G}$ , then*

$$(3.2) \quad \sum_{v \in V(\mathcal{G})} q^{\alpha(v)} Q_{\sigma(v)}(X) = \sum_{\lambda} \sum_{C \in C(\lambda)} q^{\alpha(C)} s_{\lambda}(X).$$

*In particular, the generating function for  $\mathcal{G}$  so defined is symmetric and Schur positive.*

We can, of course, include multivariate statistics in (3.2), but as our immediate purpose is to apply this theory to LLT polynomials, a single parameter suffices.

Equation 3.2 appears to be difficult to work with since, in general, it is difficult to determine when two signed, colored graphs are isomorphic. However, this problem simplifies for dual equivalence graphs. For each vertex  $v$  of a dual equivalence graph, let  $\pi(v)$  be the composition formed by the lengths of the runs of the  $+1$ 's in  $\sigma(v)$ . As shown in Proposition 3.11, each  $\mathcal{G}_\lambda$  contains a unique vertex  $T_\lambda$  with the property that  $\pi(T_\lambda)$  forms a partition and, if  $\pi(T)$  also forms a partition for some  $T \in \text{SYT}(\lambda)$ , then  $\pi(T) \leq \pi(T_\lambda)$  in dominance order. Therefore if we know which vertices occur on a given connected component of a dual equivalence graph, determining the  $\mathcal{G}_\lambda$  to which the component is isomorphic is simply a matter of comparing  $\pi(v)$  for each vertex of the component.

**3.3. The structure of dual equivalence graphs.** We begin the proof of Theorem 3.9 by showing that the standard dual equivalence graphs are non-redundant in the sense that they are mutually non-isomorphic and have no nontrivial automorphisms. Both results stem from the observation that  $\mathcal{G}_\lambda$  contains a unique vertex such that the composition formed by the lengths of the runs of  $+1$ 's in the signature gives a maximal partition.

**Proposition 3.11.** *If  $\phi : \mathcal{G}_\lambda \rightarrow \mathcal{G}_\mu$  is an isomorphism, then  $\lambda = \mu$  and  $\phi = \text{id}$ .*

*Proof.* Let  $T_\lambda$  be the tableau obtained by filling the numbers 1 through  $n$  into the rows of  $\lambda$  from left to right, bottom to top, in which case  $\sigma(T_\lambda) = +^{\lambda_1-1}, -, +^{\lambda_2-1}, -, \dots$ . For any standard tableau  $T$  such that  $\sigma(T) = \sigma(T_\lambda)$ , the numbers 1 through  $\lambda_1$ , and also  $\lambda_1 + 1$  through  $\lambda_1 + \lambda_2$ , and so on, must form horizontal strips. In particular, if  $\sigma(T) = \sigma(T_\lambda)$  for some  $T$  of shape  $\mu$ , then  $\lambda \leq \mu$  with equality if and only if  $T = T_\lambda$ .

Suppose  $\phi : \mathcal{G}_\lambda \rightarrow \mathcal{G}_\mu$  is an isomorphism. Let  $T_\lambda$  be as above for  $\lambda$ , and let  $T_\mu$  be the corresponding tableau for  $\mu$ . Then since  $\sigma(\phi(T_\lambda)) = \sigma(T_\lambda)$ ,  $\lambda \leq \mu$ . Conversely, since  $\sigma(\phi^{-1}(T_\mu)) = \sigma(T_\mu)$ ,  $\mu \leq \lambda$ . Therefore  $\lambda = \mu$ . Furthermore,  $\phi(T_\lambda) = T_\lambda$ . For  $T \in \text{SYT}(\lambda)$  such that  $\{T_\lambda, T\} \in E_i$ , we have  $\{T_\lambda, \phi(T)\} \in E_i$ , so  $\phi(T) = T$  by dual equivalence axiom 1. Extending this, every tableau connected to a fixed point by some sequence of edges is also a fixed point for  $\phi$ , hence  $\phi = \text{id}$  on each  $\mathcal{G}_\lambda$  by Proposition 3.2.  $\square$

In order to avoid cumbersome notation, as we investigate the connection between an arbitrary dual equivalence graph and the standard one, we will often abuse notation by simultaneously referring to  $\sigma$  and  $E$  as the signature function and edge set for both graphs.

**Definition 3.12.** Let  $\mathcal{G} = (V, \sigma, E)$  be a signed, colored graph of type  $(n, N)$  satisfying axiom 1. For  $1 < i < N$ , we say that a vertex  $w \in V$  admits an  $i$ -neighbor if  $\sigma(w)_{i-1} = -\sigma(w)_i$ .

For  $1 < i < n$ , if  $\sigma(w)_{i-1} = -\sigma(w)_i$  for some  $w \in V$ , then axiom 1 implies the existence of  $x \in V$  such that  $\{w, x\} \in E_i$ . That is, if  $w$  admits an  $i$ -neighbor for some  $1 < i < n$ , then  $w$  has an  $i$ -neighbor in  $\mathcal{G}$ . For  $n \leq i < N$ , though  $i$ -edges do not exist in  $\mathcal{G}$ , if  $\mathcal{G}$  were the restriction of a graph of type  $(i+1, N)$  also satisfying axiom 1, then the condition  $\sigma(w)_{i-1} = -\sigma(w)_i$  would imply the existence of a vertex  $x$  such that  $\{w, x\} \in E_i$  in the type  $(i+1, N)$  graph. When convenient,  $E_i$  may be regarded as an involution on vertices admitting an  $i$ -neighbor, i.e. if  $w$  admits an  $i$ -neighbor, then  $E_i(w) = x$  where  $\{w, x\} \in E_i$ .

Recall the notion of augmenting a partition  $\lambda$  by a skew tableau  $A$  and the resulting dual equivalence graph  $\mathcal{G}_{\lambda, A}$  from Remark 3.6.

**Lemma 3.13.** *Let  $\mathcal{G} = (V, \sigma, E)$  be a connected dual equivalence graph of type  $(n, N)$ , and let  $\phi$  be an isomorphism from the  $(n, n)$ -restriction of  $\mathcal{G}$  to  $\mathcal{G}_\lambda$  for some partition  $\lambda$  of  $n$ . Then there exists a semi-standard tableau  $A$  of shape  $\rho/\lambda$ ,  $|\rho| = N$ , with entries  $n+1, \dots, N$  such that  $\phi$  gives an isomorphism from  $\mathcal{G}$  to  $\mathcal{G}_{\lambda, A}$ . Moreover, the position of the cell of  $A$  containing  $n+1$  is unique.*

*Proof.* By axiom 2 and the fact that  $\mathcal{G}$  is connected,  $\sigma_h$  is constant on  $\mathcal{G}$  for  $h \geq n+1$ . Therefore once a suitable cell for  $n+1$  has been chosen, the cells for  $n+2, \dots, N$  may be chosen in any way that gives the correct signature. One solution is to place  $j$  north of the first column if  $\sigma_{j-1} = -1$  or east of the first row if  $\sigma_{j-1} = +1$  for  $j = n+2, \dots, N$ . Assume, then, that  $N = n+1$ .

By dual equivalence axiom 2,  $\sigma_n$  is constant on connected components of the  $(n-1, n+1)$ -restriction of  $\mathcal{G}$ . By Proposition 3.2, a connected component of the  $(n-1, n-1)$ -restriction of  $\mathcal{G}_\lambda$  consists of all standard Young tableaux where  $n$  lies in a particular northeastern cell of  $\lambda$ . Therefore, for each connected component of the  $(n-1, n+1)$ -restriction of  $\mathcal{G}$ , we may identify its image under  $\phi$  with  $\mathcal{G}_\mu$  for some partition  $\mu \subset \lambda$ ,  $|\mu| = n-1$ , with  $n$  lying in position  $\lambda/\mu$ . We will show that  $\sigma_n$  has the monotonicity property on connected components of the  $(n-1, n+1)$ -restriction of  $\mathcal{G}$  depicted in Figure 9, i.e., there is an inner corner above which  $\sigma_n = +1$  and below which  $\sigma_n = -1$ .

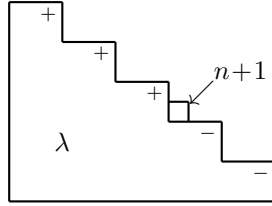


FIGURE 9. Identifying the unique position for  $n+1$  based on  $\sigma_n$ .

Let  $\mathcal{C}$  and  $\mathcal{D}$  be two distinct connected components of the  $(n-1, n+1)$ -restriction of  $\mathcal{G}$  such that there exist vertices  $v$  of  $\mathcal{C}$  and  $u$  of  $\mathcal{D}$  with  $\{v, u\} \in E_{n-1}$ . Let  $\phi(\mathcal{C}) \cong \mathcal{G}_\mu$ , and let  $\phi(\mathcal{D}) \cong \mathcal{G}_\nu$ . Since  $\{v, u\} \in E_{n-1}$ ,  $\phi(v)$  must have  $n-1$  in position  $\lambda/\mu$  with  $n-2$  lying between  $n-1$  and  $n$  in the content reading word. Since  $\phi$  preserves  $E_{n-1}$  edges,  $\phi(u)$  must be the result of an elementary dual equivalence on  $\phi(v)$  for  $n-2, n-1, n$ , which will necessarily interchange  $n-1$  and  $n$ . Since  $\phi$  preserves signatures,  $\lambda/\nu$  lies northwest of the position of  $\lambda/\mu$  if and only if  $\sigma(v)_{n-2, n-1} = + -$  and  $\sigma(u)_{n-2, n-1} = - +$ . If  $\lambda/\nu$  lies northwest of the position of  $\lambda/\mu$  and  $\sigma(v)_n = -1$ , then that  $\sigma(v)_n = \sigma(v)_{n-1}$ . Thus, by axiom 3,  $\sigma(u)_n = \sigma(v)_n = -1$ . Similarly, if  $\lambda/\nu$  lies northwest of the position of  $\lambda/\mu$  and  $\sigma(u)_n = +1$ , then  $\sigma(u)_n = \sigma(u)_{n-1}$ . Thus, by axiom 3,  $\sigma(v)_n = \sigma(u)_n = +1$ .

Abusing notation and terminology, we have shown that if  $\sigma_n(\mathcal{C}) = +1$  and  $\mathcal{D}$  is any component connected to  $\mathcal{C}$  by an  $n-1$ -edge such that  $\phi(\mathcal{D})$  lies northwest of  $\phi(\mathcal{C})$ , then  $\sigma_n(\mathcal{D}) = +1$  as well. Similarly, if  $\sigma_n(\mathcal{C}) = -1$  and  $\mathcal{D}$  is any component connected to  $\mathcal{C}$  by an  $n-1$ -edge such that  $\phi(\mathcal{D})$  lies southeast of  $\phi(\mathcal{C})$ , then  $\sigma_n(\mathcal{D}) = -1$  as well. By dual equivalence graph axiom 6, for any two distinct connected components  $\mathcal{C}$  and  $\mathcal{D}$  of the  $(n-1, n+1)$ -restriction of  $\mathcal{G}$  and any pair of vertices  $w$  on  $\mathcal{C}$  and  $x$  on  $\mathcal{D}$ , there is a path from  $w$  to  $x$  crossing at most one, and hence exactly one,  $n-1$  edge. Therefore for any  $\mathcal{C}$  and  $\mathcal{D}$ , there exist vertices  $v$  of  $\mathcal{C}$  and  $u$  of  $\mathcal{D}$  such that  $\{v, u\} \in E_{n-1}$ . Hence every two connected components of the  $(n-1, n+1)$ -restriction of  $\mathcal{G}$  are connected by an  $n-1$ -edge, thus establishing the monotonicity depicted in Figure 9.

This established, it follows that there exists a unique row such that  $\sigma(\mathcal{C})_n = -1$  whenever the  $\phi(\mathcal{C})$  has  $n$  south of this row and  $\sigma(\mathcal{C})_n = +1$  whenever the  $\phi(\mathcal{C})$  has  $n$  north of this row. In this case, the cell containing  $n+1$  must be placed at the eastern end of this pivotal row, and doing so extends  $\phi$  to an isomorphism between  $(n, n+1)$  graphs.  $\square$

Once Theorem 3.9 has been proved, Lemma 3.13 may be used to obtain the following generalization of Theorem 3.9 for dual equivalence graphs of type  $(n, N)$ : Every connected component of a

dual equivalence graph of type  $(n, N)$  is isomorphic to  $\mathcal{G}_{\lambda, A}$  for a unique partition  $\lambda$  and some skew tableau  $A$  of shape  $\rho/\lambda$ ,  $|\rho| = N$ , with entries  $n+1, \dots, N$ .

Finally we have all of the ingredients necessary to prove the main result of this section.

**Theorem 3.14.** *Let  $\mathcal{G}$  be a connected signed, colored graph of type  $(n+1, n+1)$  satisfying axioms 1 through 5 such that each connected component of the  $(n, n)$ -restriction of  $\mathcal{G}$  is isomorphic to a standard dual equivalence graph. Then there exists a morphism  $\phi$  from  $\mathcal{G}$  to  $\mathcal{G}_\lambda$  for some unique partition  $\lambda$  of  $n+1$ .*

*Proof.* When  $n+1 = 2$  or, more generally, when  $\mathcal{G}$  has no  $n$ -edges, the result follows immediately from Lemma 3.13. Therefore we proceed by induction, assuming that  $\mathcal{G}$  has at least one  $n$ -edge and assuming the result for graphs of type  $(n, n)$ .

By induction, for every connected component  $\mathcal{C}$  of the  $(n, n+1)$ -restriction of  $\mathcal{G}$ , we have an isomorphism from the  $(n, n)$ -restriction of  $\mathcal{C}$  to  $\mathcal{G}_\mu$  for a unique partition  $\mu$  of  $n$ . By Lemma 3.13, this isomorphism extends to an isomorphism from  $\mathcal{C}$  to  $\mathcal{G}_{\mu, A}$  for a unique augmenting tableau  $A$ , say with shape  $\lambda/\mu$ . We will show that for any  $\mathcal{C}$  the shape of  $\mu$  augmented with  $A$  is the same and that we may glue these isomorphisms together to obtain a morphism from  $\mathcal{G}$  to  $\mathcal{G}_\lambda$ .

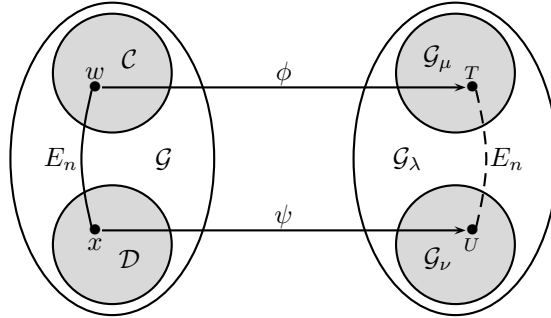
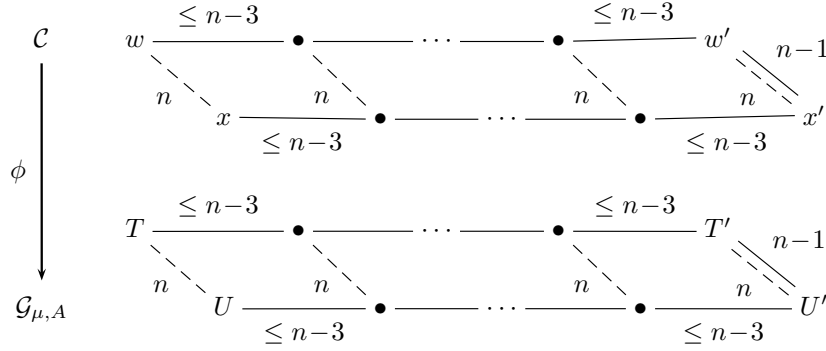


FIGURE 10. An illustration of the gluing process.

Suppose  $\{w, x\} \in E_n$ . Let  $\mathcal{C}$  (resp.  $\mathcal{D}$ ) denote the connected component of the  $(n, n+1)$ -restriction of  $\mathcal{G}$  containing  $w$  (resp.  $x$ ). Let  $\phi$  (resp.  $\psi$ ) be the isomorphism from  $\mathcal{C}$  (resp.  $\mathcal{D}$ ) to  $\mathcal{G}_{\mu, A}$  (resp.  $\mathcal{G}_{\nu, B}$ ), and set  $T = \phi(w)$ ; see Figure 10. We will show that  $\psi(x) = E_n(T)$ , and hence if  $\mu, A$  has shape  $\lambda$ , then so does  $\nu, B$  and the maps  $\phi$  and  $\psi$  glue together to give an morphism from  $\mathcal{C} \cup \mathcal{D}$  to  $\mathcal{G}_\lambda$  that preserves  $n$ -edges. There are two cases to consider, based on the relative positions of  $n-1, n, n+1$  in  $T$ , regarded as a tableau of shape  $\lambda$ .

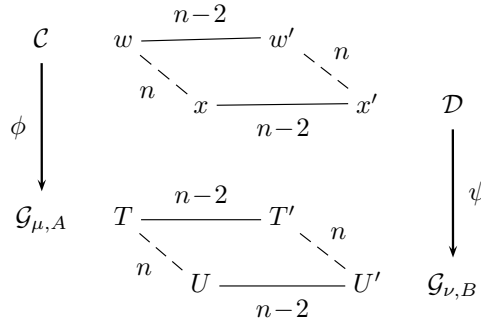
First suppose that  $n+1$  lies between  $n$  and  $n-1$  in the reading word of  $T$ . We will show that, in this case,  $\mathcal{C} = \mathcal{D}$ . Since  $n+1$  lies between  $n$  and  $n-1$  in the reading word of  $T$ , both  $n-1$  and  $n$  must be northeastern corners, and so there is a cell with entry less than  $n-1$  that also lies between them. By Proposition 3.2, there exists a tableau  $T'$  with  $n-1, n, n+1$  in the same positions as in  $T$ , but now with  $n-2$  lying between  $n$  and  $n-1$  in the reading word of  $T'$ . Furthermore, since both  $T$  and  $T'$  lie on the  $(n-2, n+1)$ -restriction of  $\mathcal{G}_{\mu, A}$ , there is a path from  $T$  to  $T'$  in  $\mathcal{G}_{\mu, A}$  using only edges  $E_h$  with  $h \leq n-3$ . Let  $U' = E_n(T')$ . Then since  $n-2$  lies between  $n$  and  $n-1$  in  $U'$ , we have  $U' = E_{n-1}(T')$  as well. By axioms 2 and 5, all edges in the path from  $T$  to  $T'$  commute with  $E_n$ , and so the same path takes  $U = E_n(T)$  to  $U'$ , and each pair of corresponding tableaux on the two paths is connected by an  $E_n$  edge; see Figure 11.

Since the path from  $T$  to  $T'$  to  $U'$  to  $U$  uses only edges from  $\mathcal{G}_{\mu, A}$ , this path lifts via the isomorphism  $\phi$  to a path in  $\mathcal{C}$ . Let  $w' = \phi^{-1}(T')$  and  $x' = \phi^{-1}(U')$ . We will show that  $x = \phi^{-1}(U)$  and so lies on  $\mathcal{C}$ . Since  $\phi$  preserves signatures, both  $w'$  and  $x'$  must admit an  $n$ -edge in  $\mathcal{G}$ . As summarized in Figure 7, axioms 3 and 4 dictate that the only way for two vertices connected by an  $n-1$ -edge both to admit an  $n$ -edge is for  $\{w', x'\} \in E_n$  in  $\mathcal{G}$ . By axioms 2 and 5, the path from  $w'$

FIGURE 11. Illustration of the path from  $T$  to  $U$  in  $\mathcal{G}_{\mu,A}$  and its lift in  $\mathcal{C}$ .

to  $w$  gives an identical path from  $x'$  to  $\phi^{-1}(U)$ . Since each corresponding pair along the two paths must be paired by an  $n$ -edge, we have  $\phi^{-1}(U) = E_n(w) = x$ , as desired. Therefore  $x$  lies on  $\mathcal{C}$ , and  $\phi$  respects the  $n$ -edge between  $w$  and  $x$ . In this case  $\mathcal{C} = \mathcal{D}$  and, by Proposition 3.11,  $\psi = \phi$ .

For the second case, suppose that  $n-1$  lies between  $n$  and  $n+1$  in  $T$ . Consider the subset of tableaux in  $\mathcal{G}_{\mu,A}$  with  $n$  and  $n+1$  fixed in the same position as in  $T$  and  $n-1$  lying anywhere between them in the reading word. In terms of the graph structure, these are all tableaux reachable from  $T$  using edges  $E_h$  with  $h \leq n-3$  and a certain subset of the  $E_{n-2}$  edges. We will return soon to the question of which  $E_{n-2}$  edges these are. For now, let  $\mathcal{T}$  denote the union of the graphs  $\mathcal{G}_{\rho,R}$ , where  $\rho$  is a partition of  $n-2$  with augmenting tableau  $R$  consisting of a single cell containing  $n-1$  such that  $\rho, R$  is the shape of  $T$  with  $n$  and  $n+1$  removed and the augmented cell of  $R$  lies strictly between the positions of  $n$  and  $n+1$  in  $T$ . Clearly the set of  $\rho, R$  uniquely determines the cells containing  $n$  and  $n+1$ , and so uniquely determines  $\lambda$ . Furthermore, which of  $n, n+1$  occupies which cell is determined by  $\sigma_n$ , which is constant on this subset by axioms 2 and 3. Lifting  $\mathcal{T}$  to  $\mathcal{C}$  using  $\phi^{-1}$  gives rise to an induced subgraph of  $\mathcal{C}$  that completely determines  $\lambda$  as well as the positions of  $n$  and  $n+1$  in the image of this subgraph under  $\phi$ . We will show that the corresponding induced subgraph for  $\mathcal{D}$  is isomorphic but with the opposite sign for  $\sigma_n$ .

FIGURE 12. Illustration of  $E_{n-2}$  edges on  $\mathcal{T} \cup \mathcal{U}$  and their lifts in  $\mathcal{C} \cup \mathcal{D}$ .

To prove the assertion, we return to the question of which  $E_{n-2}$  edges are allowed in generating  $\mathcal{T}$ . Any  $E_{n-2}$  edge that keeps  $n-1$  between  $n$  and  $n+1$  clearly does not change  $\sigma_{n-1}$  or  $\sigma_n$ . Therefore such  $E_{n-2}$  edges must pair vertices both of which admit an  $n$ -neighbor. Further, neither of these vertices may have  $E_n$  as a double edge with  $E_{n-1}$  since  $n-1$  lies between  $n$  and  $n+1$ . By axiom 4, the  $E_{n-2}$  edges that meet these conditions are precisely those in the lower component of Figure 8. In particular, these  $E_{n-2}$  edges commute with  $E_n$  edges as depicted in Figure 12. By axioms 2 and 5,  $E_h$  also commutes with  $E_n$  for  $h \leq n-3$ . Therefore all edges on the induced subgraph of  $\mathcal{C}$

containing  $\phi^{-1}(\mathcal{T})$  commute with  $E_n$ . Therefore  $E_n$  may be regarded as an isomorphism from this subgraph to  $\mathcal{X} = E_n(\phi^{-1}(\mathcal{T}))$ . Since  $\{w, x\} \in E_n$  and  $w \in \phi^{-1}(\mathcal{T})$ , we have  $x \in \mathcal{X}$ . Since all edges of the induced subgraph have color at most  $n-2$ , it follows that  $\mathcal{X} \subset \mathcal{D}$ .

Let  $U = \psi(x)$ , and, more generally, let  $\mathcal{U} = \psi(\mathcal{X})$ . Since  $\phi, \psi$  and  $E_n$  are isomorphisms,  $\mathcal{U}$  together with its induced edges is isomorphic to  $\mathcal{T}$  together with its induced edges, though, by axiom 1, the signs for  $\sigma_n$  and  $\sigma_{n+1}$  are reversed. By the earlier characterization of  $\mathcal{T}$ , this implies that the tableaux in  $\mathcal{U}$  have shape  $\lambda$ , with the cells containing  $n$  and  $n+1$  reversed from that in  $\mathcal{T}$ . In particular,  $\mathcal{U} = E_n(\mathcal{T})$ , that is to say,  $\phi$  and  $\psi$  glue to give a morphism from  $\mathcal{C} \cup \mathcal{D} \subset \mathcal{G}$  to  $\mathcal{G}_{\mu,A} \cup \mathcal{G}_{\nu,B} \subset \mathcal{G}_\lambda$  that respects  $E_n$  edges of the induced subgraphs.

Since  $T$  admits an  $n$ -neighbor,  $n$  cannot lie between  $n-1$  and  $n+1$ , so these two are the only cases. Thus we now have a well-defined morphism from the  $(n, n+1)$ -restriction of  $\mathcal{G}$  to the  $(n, n+1)$ -restriction of  $\mathcal{G}_\lambda$  that respects  $n$ -edges. As such, this map lifts to a morphism from  $\mathcal{G}$  to  $\mathcal{G}_\lambda$ .  $\square$

By Remark 3.8, the morphism of Theorem 3.14 is necessarily surjective, though in general it need not be injective. The smallest example where injectivity fails was first observed by Gregg Musiker in a graph of type  $(6, 6)$  with generating function  $2s_{(3,2,1)}(X)$ ; see Figure 43 in Appendix B.

**Corollary 3.15.** *Let  $\mathcal{G}$  satisfy the hypotheses of Theorem 3.14. Then the fiber over each vertex of  $\mathcal{G}_\lambda$  in the morphism from  $\mathcal{G}$  to  $\mathcal{G}_\lambda$  has the same cardinality.*

*Proof.* Let  $\phi$  be the morphism from  $\mathcal{G}$  to  $\mathcal{G}_\lambda$ . We show that for any connected component  $\mathcal{C}$  of the  $(n, n)$ -restriction of  $\mathcal{G}$ , say with  $\phi(\mathcal{C}) = \mathcal{G}_\mu$ , and any partition  $\nu \subset \lambda$  of size  $n$ , there is a unique connected component  $\mathcal{D}$  of the  $(n, n)$ -restriction of  $\mathcal{G}$  with  $\phi(\mathcal{D}) = \mathcal{G}_\nu$  that can be reached from  $\mathcal{C}$  by crossing at most one  $E_n$  edge. Once established, this gives a bijective correspondence between connected components of  $\phi^{-1}(\mathcal{G}_\mu)$  and connected components of  $\phi^{-1}(\mathcal{G}_\nu)$ , thus proving the result.

To prove existence, if  $\nu \neq \mu$ , let  $T$  be a tableau of shape  $\lambda$  with  $n+1$  in position  $\lambda/\mu$ ,  $n$  in position  $\lambda/\nu$ , and  $n-1$  lying between in the reading word. Otherwise let  $T$  be a tableau with  $n+1$  in position  $\lambda/\mu$  and  $n$  and  $n-1$  lying on opposite sides in the reading word. Let  $w$  be the unique element in  $\phi^{-1}(T) \cap \mathcal{C}$ . Then  $w$  admits an  $n$ -neighbor, and, since  $\phi$  is a morphism,  $\phi(E_n(w)) = E_n(\phi(w)) \in \mathcal{G}_\nu$ .

To prove uniqueness, let  $\{w, x\} \in E_n$  with  $w \in \mathcal{C} \cong \mathcal{G}_\mu$  and  $x \in \mathcal{D} \cong \mathcal{G}_\nu$ . If  $n+1$  lies between  $n$  and  $n-1$  in  $\phi(w)$ , then  $\mu = \nu$ , and just as in the proof of Theorem 3.14, we concluded that  $\mathcal{D} = \mathcal{C}$  as desired. Alternately, assume  $n-1$  lies between  $n$  and  $n+1$  in  $\phi(w)$ , and suppose  $\{w', x'\} \in E_{n-1}$  with  $w' \in \mathcal{C}$  and  $x' \in \mathcal{D}' \cong \mathcal{G}_\nu$ . Since  $\phi(w)$  and  $\phi(w')$  have the same shape, and  $E_n(\phi(w)) = \phi(E_n(w)) = \phi(x)$  and  $E_n(\phi(w')) = \phi(E_n(w')) = \phi(x')$  have the same shape, just as in the proof of Theorem 3.14, there must be a path from  $\phi(w)$  to  $\phi(w')$  in  $\mathcal{G}_\nu$  using only edges  $E_h$  with  $h \leq n-3$  and those  $E_{n-2}$  that commute with  $E_n$ . Therefore this path gives rise to the same path from  $\phi(x)$  to  $\phi(x')$  in  $\mathcal{G}_\mu$ . The former path lifts to a path from  $w$  to  $w'$  in  $\mathcal{C}$ , and so the latter lifts to a path from  $E_n(w) = x$  to  $E_n(w') = x'$  in  $\mathcal{D} = \mathcal{D}'$ , which is as desired.  $\square$

In order to ensure that the morphism in the conclusion of Theorem 3.14 is an isomorphism, and thereby complete the proof of Theorem 3.9, we need only invoke the heretofore uninvoked axiom 6.

*Proof of Theorem 3.9.* Let  $\mathcal{G}$  be a dual equivalence graph of type  $(n+1, n+1)$ . We aim to show that  $\mathcal{G}$  is isomorphic to  $\mathcal{G}_\lambda$  for a unique partition  $\lambda$  of  $n+1$ . We proceed by induction on  $n+1$ , noting that the result is trivial for  $n+1 = 2$ . Every connected component of the  $(n, n)$ -restriction of  $\mathcal{G}$  is a dual equivalence graph, and so, by induction, is isomorphic to a standard dual equivalence graph. Thus, by Theorem 3.14, there exists a morphism, say  $\phi$ , from  $\mathcal{G}$  to  $\mathcal{G}_\lambda$  for a unique partition  $\lambda$  of  $n+1$ . By Corollary 3.15, for any connected component  $\mathcal{C}$  of the  $(n, n)$ -restriction of  $\mathcal{G}$  and any partition  $\nu \subset \lambda$  of size  $n$ , there is a unique connected component  $\mathcal{D}$  of the  $(n, n)$ -restriction of  $\mathcal{G}$  that can be reached from  $\mathcal{C}$  by crossing at most one  $E_n$  edge such that  $\phi(\mathcal{D}) = \mathcal{G}_\nu$ . By dual equivalence axiom 6, any two connected components of the  $(n, n)$ -restriction of  $\mathcal{G}$  can be connected by a path using at most one  $E_n$  edge. Therefore the connected components of the  $(n, n)$ -restriction of  $\mathcal{G}$  are pairwise non-isomorphic. Hence the morphism from  $\mathcal{G}$  to  $\mathcal{G}_\lambda$  is injective on the  $(n, n+1)$ -restrictions, and so it is injective on all of  $\mathcal{G}$ . Surjectivity follows from Remark 3.8, thus  $\phi$  is an isomorphism.  $\square$



## 4. A GRAPH FOR LLT POLYNOMIALS

**4.1. Words with content.** In this section we describe a modified characterization of LLT polynomials as the generating function of  $k$ -ribbon words. As Proposition 4.2 shows, these are precisely the content reading words of semi-standard  $k$ -tuples of tableaux.

Given a word  $w$  and a non-decreasing sequence of integers  $c$  of the same length, define the  $k$ -descent set of the pair  $(w, c)$ , denoted  $\text{Des}_k(w, c)$ , by

$$(4.1) \quad \text{Des}_k(w, c) = \{(i, j) \mid w_i > w_j \text{ and } c_j - c_i = k\}.$$

**Definition 4.1.** A  $k$ -ribbon word is a pair  $(w, c)$  consisting of a word  $w$  and a non-decreasing sequence of integers  $c$  of the same length such that if  $c_i = c_{i+1}$ , then there exist integers  $h$  and  $j$  such that  $(h, i), (i+1, j) \in \text{Des}_k(w, c)$  and  $(i, j), (h, i+1) \notin \text{Des}_k(w, c)$ . In other words,  $c_h = c_i - k$  and  $w_i < w_h \leq w_{i+1}$  while  $c_j = c_i + k$  and  $w_i \leq w_j < w_{i+1}$ .

**Proposition 4.2.** The pair  $(w, c)$  is a  $k$ -ribbon word if and only if there exists a  $k$ -tuple of (skew) semi-standard tableaux such that  $w$  is the content reading word of the  $k$ -tuple and  $c$  gives the corresponding contents.

*Proof.* Suppose first that  $w$  is the content reading word of some  $k$ -tuple of semi-standard tableaux with corresponding shifted contents given by  $c$ . If  $c_i = c_{i+1}$ , then in the  $k$ -tuple there must exist entries  $w_h$  and  $w_j$  as shown in Figure 13. The semi-standard condition ensures that  $w_i < w_h \leq w_{i+1}$  and  $w_i \leq w_j < w_{i+1}$ . Therefore the conditions of Definition 4.1 are met.

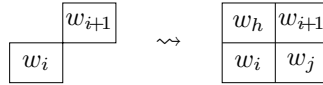


FIGURE 13. Situation when  $c_i = c_{i+1}$  for a  $k$ -tuple of semi-standard Young tableaux.

Now suppose that  $(w, c)$  is a  $k$ -ribbon word. For each  $j$ , arrange all  $w_i$  such that  $c_i = j$  into cells along a southwest to northeast diagonal in increasing order. Align the southwest corner of the diagonal for  $j - k$  immediately north (resp. west) of the southwest corner of the diagonal for  $j$  whenever the smallest letter with content  $j - k$  is greater than (resp. less than or equal to) the smallest letter with content  $j$ .

We must show that the result is a  $k$ -tuple of (skew) shapes whose entries satisfy the semi-standard condition. Consider two adjacent diagonals  $j - k$  and  $j$ . By construction, the southwestern most cells of the diagonals form a partition shape and satisfy the semi-standard condition. By induction, assume that the entries in diagonal  $j - k$  through  $w_h$  and the entries in diagonal  $j$  through  $w_i$  belong to a semi-standard tableau of skew shape, with  $w_h$  immediately west or immediately north of  $w_i$ .

Suppose that  $c_{i+1} = c_i$ , noting that the case when  $c_{h+1} = c_h$  may be solved similarly. If  $w_h > w_i$ , then we must show that  $w_h \leq w_{i+1}$ . By Definition 4.1, there exists an integer  $l$  such that  $(l, i+1) \notin \text{Des}_k(w, c)$ , and therefore  $w_l \leq w_{i+1}$ . Since  $c_l = j - k$ , we have  $w_h \leq w_l \leq w_{i+1}$ . If  $w_h \leq w_i$ , then we must show that  $c_{h+1} = j - k$  and  $w_i < w_{h+1} \leq w_{i+1}$ . By Definition 4.1, there exists an integer  $l$  such that  $(l, i) \in \text{Des}_k(w, c)$  and  $(l, i+1) \notin \text{Des}_k(w, c)$ . Therefore  $c_l = j - k$  and  $w_h \leq w_i < w_l \leq w_{i+1}$ . The non-decreasing condition on  $c$  implies that  $c_{h+1} = j - k$ , and so there exists an integer  $m$  such that  $(h+1, m) \in \text{Des}_k(w, c)$  and  $(h, m) \notin \text{Des}_k(w, c)$ , i.e.  $w_h \leq w_m < w_{h+1}$  with  $c_m = j$ . The only way to satisfy these two conditions is to have  $m = i$  and  $l = h+1$ .  $\square$

For  $\mathbf{T}$  and  $\mathbf{U}$  two  $k$ -tuples of semi-standard tableaux, let  $(w_{\mathbf{T}}, c_{\mathbf{T}})$  and  $(w_{\mathbf{U}}, c_{\mathbf{U}})$  denote the corresponding  $k$ -ribbon words. Then  $\mathbf{T}$  and  $\mathbf{U}$  have the same shape if and only if  $\text{Des}_k(w_{\mathbf{T}}) = \text{Des}_k(w_{\mathbf{U}})$  and  $c_{\mathbf{T}} = c_{\mathbf{U}}$ . In particular, if we let  $\text{WRib}_k(c, D)$  denote the set of  $k$ -ribbon words with content vector  $c$  and  $k$ -descent set  $D$ , then we have established a bijective correspondence

$$(4.2) \quad \text{WRib}_k(c, D) \xleftrightarrow{\sim} \text{SSYT}_k(\mu).$$

Define the *set of  $k$ -inversions* and the  *$k$ -inversion number* of a pair  $(w, c)$  by

$$\begin{aligned} \text{Inv}_k(w, c) &= \{(i, j) \mid w_i > w_j \text{ and } k > c_j - c_i > 0\}, \\ \text{inv}_k(w, c) &= |\text{Inv}_k(w, c)|. \end{aligned}$$

Recalling (2.7), we have

$$(4.3) \quad \text{Inv}_k(w_{\mathbf{T}}, c_{\mathbf{T}}) = \text{Inv}_k(\mathbf{T}).$$

Therefore we may express LLT polynomials in terms of  $k$ -ribbon words as follows.

**Corollary 4.3.** *Let  $\mu$  be a (skew) shape, and let  $c, D$  be the content vector and  $k$ -descent set corresponding to  $\mu$  by (4.2). Then*

$$(4.4) \quad \tilde{G}_{\mu}^{(k)}(x; q) = \sum_{(w, c) \in \text{WRib}_k(c, D)} q^{\text{inv}_k(w, c)} x^w = \sum_{\substack{(w, c) \in \text{WRib}_k(c, D) \\ w \text{ a permutation}}} q^{\text{inv}_k(w, c)} Q_{\sigma(w)}(x),$$

where  $x^w$  is the monomial  $x_1^{\pi_1} x_2^{\pi_2} \cdots$  when  $w$  has weight  $\pi$ , and  $\sigma(w)$  is defined as in (2.4).

**4.2. Dual equivalence for tuples of tableaux.** Let  $V_{c, D}^{(k)}$  denote the set of permutations  $w$  such that  $(w, c)$  is a standard  $k$ -ribbon word with  $k$ -descent set  $D$ , i.e.

$$(4.5) \quad V_{c, D}^{(k)} = \{w \mid (w, c) \text{ is a standard } k\text{-ribbon word with } \text{Des}_k(w, c) = D\}.$$

Define the distance between two letters  $i$  and  $j$  of  $w \in V_{c, D}^{(k)}$  by

$$(4.6) \quad \text{dist}(w_i, w_j) = |c_i - c_j|,$$

with the obvious extension  $\text{dist}(a_1, \dots, a_l) = \max_{i, j} \{\text{dist}(a_i, a_j)\}$ . Note that if  $(w, c)$  is a standard  $k$ -ribbon word, then none of  $i-1, i, i+1$  may occur with the same content.

Similar to Definition 3.1, define involutions  $d_i$  and  $\tilde{d}_i$  on permutations in which  $i$  does not lie between  $i-1$  and  $i+1$  by

$$(4.7) \quad d_i(\cdots i \cdots i \pm 1 \cdots i \mp 1 \cdots) = \cdots i \mp 1 \cdots i \pm 1 \cdots i \cdots,$$

$$(4.8) \quad \tilde{d}_i(\cdots i \cdots i \pm 1 \cdots i \mp 1 \cdots) = \cdots i \pm 1 \cdots i \mp 1 \cdots i \cdots,$$

where all other entries remain fixed. Note that the former involution is precisely Haiman's dual equivalence on permutations. For fixed  $k$ , combine these two maps into an involution  $D_i^{(k)}$  by

$$(4.9) \quad D_i^{(k)}(w) = \begin{cases} d_i(w) & \text{if } \text{dist}(i-1, i, i+1) > k \\ \tilde{d}_i(w) & \text{if } \text{dist}(i-1, i, i+1) \leq k \end{cases}.$$

**Proposition 4.4.** *For  $w$  a permutation,  $c$  a content vector and  $k > 0$  an integer, we have*

$$(4.10) \quad \text{Des}_k(w, c) = \text{Des}_k(D_i^{(k)}(w), c),$$

$$(4.11) \quad \text{inv}_k(w, c) = \text{inv}_k(D_i^{(k)}(w), c).$$

*In particular,  $D_i^{(k)}$  is a well-defined involution on  $V_{c, D}^{(k)}$  that preserves the number of  $k$ -inversions.*

*Proof.* If  $i$  lies between  $i-1$  and  $i+1$  in  $w$ , then the assertion is trivial. Assume then that  $i$  does not lie between  $i-1$  and  $i+1$  in  $w$ . If  $\text{dist}(i-1, i, i+1) > k$  in  $w$ , then  $\text{Des}_k(w, c) = \text{Des}_k(d_i(w), c)$  and  $\text{Inv}_k(w, c) = \text{Inv}_k(d_i(w), c)$ . Similarly, if  $\text{dist}(i-1, i, i+1) \leq k$  in  $w$ , then  $\text{Des}_k(w, c) = \text{Des}_k(\tilde{d}_i(w), c)$  and  $\text{inv}_k(w, c) = \text{inv}_k(\tilde{d}_i(w), c)$  (though  $\text{Inv}_k(w, c) \neq \text{Inv}_k(\tilde{d}_i(w), c)$ ). The result now follows.  $\square$

For each content vector  $c$  of length  $n$ , and  $k$ -descent set  $D$ , we construct a signed, colored graph  $\mathcal{G}_{c, D}^{(k)}$  of type  $(n, n)$  on the vertex set  $V_{c, D}^{(k)}$  as follows. Define the signature function  $\sigma : V_{c, D}^{(k)} \rightarrow \{\pm 1\}^{n-1}$  by

$$(4.12) \quad \sigma(w)_i = \begin{cases} +1 & \text{if } i \text{ appears to the left of } i+1 \text{ in } w \\ -1 & \text{if } i+1 \text{ appears to the left of } i \text{ in } w \end{cases}.$$

By (4.10),  $D_i^{(k)}$  is an involution on vertices of  $V_{c,D}^{(k)}$  admitting an  $i$ -neighbor. Therefore for  $1 < i < n$ , we may define the  $i$ -colored edges  $E_i^{(k)}$  to be the set of pairs  $\{v, D_i^{(k)}(v)\}$  for each  $v$  admitting an  $i$ -neighbor. Finally, we define

$$(4.13) \quad \mathcal{G}_{c,D}^{(k)} = \left( V_{c,D}^{(k)}, \sigma, E^{(k)} \right).$$

An example of  $\mathcal{G}_{c,D}^{(k)}$  is given in Figure 14, and additional examples may be found in Appendix B.

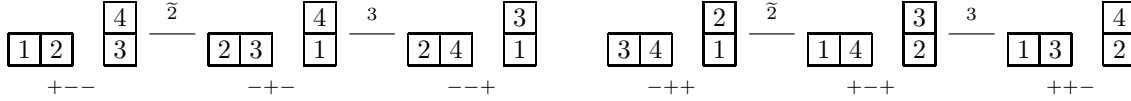


FIGURE 14. The graph  $\mathcal{G}_{(-1,0,1,2),\{(-1,1)\}}^{(2)}$  on domino tableaux of shape  $((2), (1,1))$ .

By Corollary 4.3 and (2.11), the generating function for  $\mathcal{G}_{c,D}^{(k)}$  weighted by  $\text{inv}_k(-, c)$  is given by

$$(4.14) \quad \sum_{v \in V_{c,D}^{(k)}} q^{\text{inv}_k(v, c)} Q_{\sigma(v)}(x) = \tilde{G}_{\mu}^{(k)}(x; q).$$

In particular, a formula for the Schur coefficients of the generating function for  $\mathcal{G}_{c,D}^{(k)}$  gives a formula for the Schur coefficients of the LLT polynomial  $\tilde{G}_{\mu}^{(k)}(x; q)$ . For example, since the graph in Figure 14 is a dual equivalence graph, we have

$$\tilde{G}_{(2), (1,1)}^{(2)}(x; q) = qs_{3,1}(x) + q^2 s_{2,1,1}(x).$$

In general,  $\mathcal{G}_{c,D}^{(k)}$  does not satisfy dual equivalence axioms 4 or 6; see Appendix B for examples. These graphs do, however, satisfy the other axioms as well as the following weakened version of axiom 4.

**Definition 4.5.** A signed, colored graph  $\mathcal{G} = (V, \sigma, E)$  is *locally Schur positive* if for every connected component  $\mathcal{C}$  of  $(V, \sigma, E_{i-1} \cup E_i)$  and every connected component  $\mathcal{D}$  of  $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$ , the restricted degree 4 and degree 5 generating functions

$$(4.15) \quad \sum_{v \in \mathcal{C}} Q_{\sigma(v)_{i-2, i-1, 1}}(x) \quad \text{and} \quad \sum_{v \in \mathcal{D}} Q_{\sigma(v)_{i-3, i-2, i-1, 1}}(x)$$

are symmetric and Schur positive.

Comparing Figures 7 and 8 with the standard dual equivalence graphs of sizes 4 and 5 (see Figure 6), dual equivalence graph axiom 4 implies that  $\mathcal{G}_{\lambda}$  is locally Schur positive.

**Proposition 4.6.** *For each content vector  $c$  and  $k$ -descent set  $D$ , the graph  $\mathcal{G}_{c,D}^{(k)}$  satisfies dual equivalence graph axioms 1, 2, 3 and 5, is locally Schur positive, and the  $k$ -inversion number is constant on connected components.*

*Proof.* Axiom 1 follows from the construction of  $E^{(k)}$  using (4.9), and axiom 2 can easily be seen from equations (4.7) and (4.8). For axiom 3, suppose  $\{w, x\} \in E_i^{(k)}$  and  $\sigma(w)_{i-2} = -\sigma(x)_{i-2}$ . If  $x = d_i(w)$ , then both  $i-2$  and  $i+1$  must lie between  $i-1$  and  $i$ . In particular,  $\sigma(w)_{i-2} = -\sigma(w)_{i-1}$ . If  $x = \tilde{d}_i(w)$ , then  $i-2$  must lie between the position of  $i-1$  in  $w$  and the position of  $i-1$  in  $x$ . In particular,  $i-2$  must lie between  $i-1$  and  $i$  in both  $w$  and  $x$ , and so again  $\sigma(w)_{i-2} = -\sigma(w)_{i-1}$ . The result for  $\{w, x\} \in E_i^{(k)}$  with  $\sigma(w)_{i+1} = -\sigma(x)_{i+1}$  is completely analogous. Axiom 5 follows from the fact that if  $w$  admits both an  $i$ -neighbor and a  $j$ -neighbor for some  $|i-j| \geq 3$ , then  $D_i^{(k)} D_j^{(k)}(w) = D_j^{(k)} D_i^{(k)}(w)$ .

To establish local Schur positivity, a tedious but straightforward diagram chase shows that there are exactly 25 possible non-isomorphic connected components of  $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$  in  $\mathcal{G}_{c,D}^{(k)}$ . Of

these, 7 correspond to the standard dual equivalence graphs of type  $(5, 5)$  and the remaining 18 are locally Schur positive. Alternately, note that it suffices to consider  $\mathcal{G}_{c,D}^{(k)}$  for all content vectors  $c$  of length 5 with  $c_1 = 1$  and  $c_5 \leq 25$  and all  $k \leq 5$  and all possible  $k$ -descent sets. This gives finitely many cases to computer verify. Finally, the  $k$ -inversion number is constant on connected components of  $\mathcal{G}_{c,D}^{(k)}$  by Proposition 4.4.  $\square$

As foreshadowed by Definition 4.5, the generating function of a connected component of the signed, colored graph for LLT polynomials is not, in general, a single Schur function, though it is always Schur positive. In Section 5, we describe an algorithm by which the edges of every connected component of  $\mathcal{G}_{c,D}^{(k)}$  can be transformed so that the resulting graph is indeed a dual equivalence graph. We do this inductively by constructing a sequence of signed, colored graphs

$$\mathcal{G}_{c,D}^{(k)} = \mathcal{G}_2, \dots, \mathcal{G}_{n-1} = \tilde{\mathcal{G}}_{c,D}^{(k)}$$

on the same vertex set with the same signature function with the following properties. For each  $i = 2, \dots, n-1$ , the graph  $\mathcal{G}_i$  satisfies dual equivalence graph axioms 1, 2, 3 and 5, and the  $(i+1, N)$ -restriction of  $\mathcal{G}_i$  satisfies axioms 4 and 6 (and so is a dual equivalence graph). Furthermore, vertices paired by  $E_i$  in  $\mathcal{G}_i$  have the property that they lie on the same connected component of  $(V, \sigma, E_2 \cup \dots \cup E_i)$  in  $\mathcal{G}_{i-1}$ . This construction proves the following.

**Theorem 4.7.** *For  $\mu$  a  $k$ -tuple of (skew) shapes, let  $c, D$  be the corresponding pair by (4.2), and let  $\mathcal{G}_{c,D}^{(k)}$  be the signed, colored graph constructed above. Then for every connected component  $\mathcal{C}$  of  $\mathcal{G}_{c,D}^{(k)}$ , the sum  $\sum_{v \in V(\mathcal{C})} Q_{\sigma(v)}(X)$  is symmetric and Schur positive.*

**Corollary 4.8.** *Let  $\tilde{\mathcal{G}}_{c,D}^{(k)}$  be the dual equivalence graph resulting from the transformation of the graph  $\mathcal{G}_{c,D}^{(k)}$ . Then for  $\lambda$  a partition, we have*

$$(4.16) \quad \tilde{K}_{\lambda, \mu}^{(k)}(q) = \sum_{\mathcal{C} \cong \mathcal{G}_\lambda} q^{\text{inv}_k(\mathcal{C})},$$

where the sum is taken over all connected components  $\mathcal{C}$  of  $\tilde{\mathcal{G}}_{c,D}^{(k)}$  that are isomorphic to  $\mathcal{G}_\lambda$ . In particular,  $\tilde{K}_{\lambda, \mu}^{(k)}(q) \in \mathbb{N}[q]$ , and, by (2.19),  $\tilde{K}_{\lambda, \mu}(q, t) \in \mathbb{N}[q, t]$ .

The proof of Theorem 4.7 is the content of Section 5. Before delving into the proof, we consider two extremal cases of  $\mathcal{G}_{c,D}^{(k)}$  where the connected components have particularly nice Schur expansions that can be proved by more elementary means.

**4.3. Special cases.** Since  $\text{dist}(i-1, i, i+1) \geq 2$  for every  $w \in V_{c,D}^{(k)}$ ,  $D_i^{(1)}$  is just the standard elementary dual equivalence on  $i-1, i, i+1$ . Therefore  $\mathcal{G}_{c,D}^{(1)}$  is isomorphic to the standard dual equivalence graph  $\mathcal{G}_\lambda$  for a unique partition  $\lambda$ .

When  $k \geq 3$ ,  $E_i^{(k)}$  will not give the edges of a dual equivalence graph. For instance, if  $w$  has the pattern 2431 with  $\text{dist}(1, 2, 3) \leq k$ , then  $D_2^{(k)}(w)$  contains the pattern 3412. By axiom 4, a dual equivalence graph must have  $\{w, D_2^{(k)}(w)\} \in E_2^{(k)} \cap E_3^{(k)}$ . However,  $D_2^{(k)}(w) \neq D_3^{(k)}(w)$ , so this is not the case for  $\mathcal{G}_{c,D}^{(k)}$ . Therefore for  $k \geq 3$ , Theorem 4.7 is the best we can hope for. When  $k = 2$ , however, this problematic case does not arise, and we have the following result.

**Theorem 4.9.** *The graph  $\mathcal{G}_{c,D}^{(2)}$  on 2-ribbon words with content vector  $c$  and 2-descent set  $D$  is a dual equivalence graph, and the 2-inversion number is constant on connected components.*

*Proof.* By Proposition 4.6, it suffices to show that dual equivalence axiom 4 holds. Since  $k = 2$ , if  $x = \tilde{d}_i(w)$ , then  $\sigma(w)_j = \sigma(x)_j$  for all  $j \neq i-1, i$ . In particular, if  $\{w, x\} \in E_i$  and  $\sigma(w)_{i-2} = -\sigma(x)_{i-2}$ , then  $d_i(w) = x = d_{i-1}(w)$ . This establishes axiom 4 when  $n \leq 4$ .

To prove that connected components of  $(V_{c,D}^{(2)}, \sigma, E_{i-2}^{(2)} \cup E_{i-1}^{(2)} \cup E_i^{(2)})$  have the correct form, note that it suffices to show that if  $x = D_i^{(2)}(w) = D_{i-1}^{(2)}(w)$  and  $x$  admits an  $i-2$ -neighbor, then letting  $y = D_i^{(2)}D_{i-2}^{(2)}(x)$ , we have  $D_{i-2}^{(2)}(y) = D_{i-1}^{(2)}(y)$ . In this case,  $x$  must have  $i-2$  and  $i+1$  lying between  $i$  and  $i-1$  which have contents more than 2 apart. Then in  $D_{i-2}^{(2)}(x)$ ,  $i-3, i-1$  and  $i+1$  will all lie between  $i$  and  $i-2$  which must also have contents more than 2 apart. In  $y = D_i^{(2)}D_{i-2}^{(2)}(x)$ ,  $i-3$  and  $i$  will both lie between  $i-2$  and  $i-1$  which must have contents more than 2 apart. Therefore  $D_{i-2}^{(2)}(y) = d_{i-2}(y) = d_{i-1}(y) = D_{i-1}^{(2)}(y)$ .  $\square$

Since Theorem 4.9 does not use the transformations of Section 5, we obtain a simple proof of positivity of LLT polynomials when  $k = 2$ , and also of Macdonald polynomials indexed by a partition with at most 2 columns. For a bijective proof, see also [Ass08].

Next consider the case when  $k \geq c_n - c_1$  and so  $D_i^{(k)} = \tilde{d}_i$  for all  $i$ . Now there are no double edges in  $\mathcal{G}_{c,D}^{(k)}$ . For the standard dual equivalence graphs,  $\mathcal{G}_\lambda$  has no double edges if and only if  $\lambda$  is a hook, i.e.  $\lambda = (m, 1^{n-m})$  for some  $m$ . Therefore the generating function for a dual equivalence graph with no double edges is a sum of Schur functions indexed by hooks. The analog of this fact for  $\mathcal{G}_{c,D}^{(k)}$  is that the generating function is a sum of skew Schur functions indexed by ribbons.

Let  $\nu$  be a ribbon of size  $n$ . Label the cells of  $\nu$  from 1 to  $n$  in increasing order of content. Define the *descent set* of  $\nu$ , denoted  $\text{Des}(\nu)$ , to be the set of indices  $i$  such that the cell labelled  $i+1$  lies south of the cell labelled  $i$ . Define the *major index* of a ribbon by

$$(4.17) \quad \text{maj}(\nu) = \sum_{i \in \text{Des}(\nu)} i.$$

Notice that if  $R$  is a filling of a column, and we reshape  $R$  into a semi-standard ribbon as described in Section 2.4, say of shape  $\nu$ , then (4.17) agrees with (2.13) in the sense that  $\text{maj}(\nu) = \text{maj}(R)$ .

Any connected component of  $\mathcal{G}_{c,D}^{(k)}$  such that  $D_i^{(k)} = \tilde{d}_i$  on the entire component not only has constant  $k$ -inversion number, but the relative ordering of the first and last letters of each vertex is constant as well. That is, for  $\mathcal{C}$  a connected component of  $\mathcal{G}_{c,D}^{(k)}$ ,  $w_1 > w_n$  for some  $w \in V(\mathcal{C})$  if and only if  $w_1 > w_n$  for all  $w \in V(\mathcal{C})$ . In the affirmative case, say that  $(1, n)$  is an *inversion* in  $\mathcal{C}$ .

**Theorem 4.10.** *Let  $\mathcal{G}_{c,D}^{(k)}$  be the signed, colored graph of type  $(n, n)$  on  $k$ -ribbon words with contents  $c$  and  $k$ -descent set  $D$ . Let  $\mathcal{C}$  be a connected component of  $\mathcal{G}_{c,D}$  such that  $D_i^{(k)}(v) = \tilde{d}_i(v)$  for all  $v \in V(\mathcal{C})$ . Then*

$$(4.18) \quad \sum_{v \in V(\mathcal{C})} Q_{\sigma(v)}(x) = \sum_{\nu \in \text{Rib}(\mathcal{C})} s_\nu,$$

where  $\text{Rib}(\mathcal{C})$  is the set of ribbons of length  $n$  with major index equal to  $\text{inv}_k(\mathcal{C})$  such that  $n-1$  is a descent if and only if  $(1, n)$  is an inversion in  $\mathcal{C}$ .

*Proof.* From the hypotheses on  $\mathcal{C}$ , we may assume that  $k = n$ ,  $c = (1, \dots, n)$  and  $D = \emptyset$ . Therefore  $V_{c,D}^{(k)}$  is just the set of permutations of  $[n]$  thought of as words. In this case,  $k$ -inversions are just the usual inversions for a permutation. By earlier remarks, for  $w, v \in V(\mathcal{C})$ ,  $\text{inv}(w) = \text{inv}(v)$  and  $(1, n) \in \text{Inv}(w)$  if and only if  $(1, n) \in \text{Inv}(v)$ . In fact, it is an exercise to show that this necessary condition for two vertices to coexist in  $V(\mathcal{C})$  is also sufficient. That is to say,  $V(\mathcal{C})$  is the set of words  $w$  with  $\text{inv}(w) = \text{inv}(\mathcal{C})$  and  $(1, n) \in \text{Inv}(w)$  if and only if  $(1, n)$  is an inversion of  $\mathcal{C}$ .

Recall Foata's bijection on words [Foa68]. For  $w$  a word and  $x$  a letter,  $\phi$  is built recursively using an inner function  $\gamma_x$  by  $\phi(wx) = \gamma_x(\phi(w))x$ . From this structure it follows that the last letter of  $w$  is the same as the last letter of  $\phi(w)$ . Furthermore,  $\gamma_x$  is defined so that the last letter of  $w$  is greater than  $x$  if and only if the first letter of  $\gamma_x(w)$  is greater than  $x$ , and  $\phi$  preserves the descent set of the inverse permutation, i.e.  $\sigma(w) = \sigma(\phi(w))$ . Finally, the bijection satisfies  $\text{maj}(w) = \text{inv}(\phi(w))$ .

Summarizing these properties,  $\phi$  is a  $\sigma$ -preserving bijection between the following sets:

$$\begin{aligned} \{w \mid \text{inv}(w) = j \text{ and } (1, n) \in \text{Inv}(w)\} &\xleftrightarrow{\sim} \{w \mid \text{maj}(w) = j \text{ and } n-1 \in \text{Des}(w)\}, \\ \{w \mid \text{inv}(w) = j \text{ and } (1, n) \notin \text{Inv}(w)\} &\xleftrightarrow{\sim} \{w \mid \text{maj}(w) = j \text{ and } n-1 \notin \text{Des}(w)\}. \end{aligned}$$

A standard filling of a ribbon  $\nu$  is just a permutation  $w$  such that  $\text{Des}(w) = \text{Des}(\nu)$ . Therefore by (2.5), the Schur function  $s_\nu$  may be expressed as

$$(4.19) \quad s_\nu(x) = \sum_{\text{Des}(w) = \text{Des}(\nu)} Q_{\sigma(w)}(x).$$

Applying  $\phi$  to this formula yields (4.18).  $\square$

## 5. TRANSFORMATION INTO A DUAL EQUIVALENCE GRAPH

**5.1. Packages and type.** The algorithm used to transform  $\mathcal{G}_{c,D}^{(k)}$  into a dual equivalence graph utilizes three transformations, defined in Section 5.2 and detailed in Section 5.3, that identify two  $i$ -edges on the same connected component of  $E_2 \cup \dots \cup E_i$  in  $\mathcal{G}_{i-1}$  and swap the connections in the unique way that maintains the reversal of  $\sigma_{i-1}$  and  $\sigma_i$ . For example, in Figure 15, the  $i$ -edges given by solid lines are replaced with  $i$ -edges given by the dashed lines.

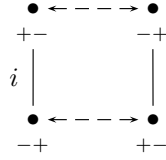


FIGURE 15. An illustration of how two  $i$ -edges are swapped in the transformation process.

By construction, this transformation preserves axiom 1. In order to maintain axioms 2 and 5, we introduce the notion of the  $i$ -package of a vertex admitting an  $i$ -neighbor. By axiom 5, if  $\{w, x\} \in E_i$  and  $\{x, y\} \in E_j$  for  $|i - j| \geq 3$ , then  $\{w, v\} \in E_j$  and  $\{v, y\} \in E_i$  for some  $v \in V$ . Changing a single  $i$ -edge may result in a violation of this condition. Therefore when one  $i$ -edge is changed, all other  $i$ -edges that subsequently violate axiom 5 must also be changed, as illustrated in Figure 16.

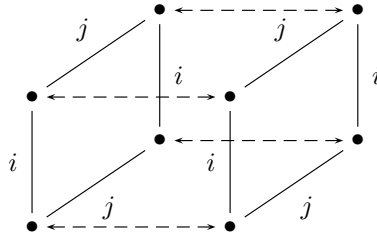


FIGURE 16. An illustration of how to maintain axiom 5 when swapping  $i$ -edges.

**Definition 5.1.** Let  $(V, \sigma, E)$  be a signed, colored graph of type  $(n, N)$  satisfying axioms 1, 2 and 5. For  $w$  a vertex of  $V$ , the  $i$ -package of  $w$  is the connected component containing  $w$  of

$$(V, (\sigma_1, \dots, \sigma_{i-3}, \sigma_{i+2}, \dots, \sigma_{N-1}), E_2 \cup \dots \cup E_{i-3} \cup E_{i+3} \cup \dots \cup E_{n-1})$$

By axiom 2, both  $\sigma_{i-1}$  and  $\sigma_i$  are constant on  $i$ -packages. Therefore  $w$  admits an  $i$ -neighbor if and only if every vertex of the  $i$ -package of  $w$  admits an  $i$ -neighbor. By axiom 5, knowing  $E_i(w)$  determines  $E_i$  on the entire  $i$ -package of  $w$ . That is to say,  $E_i$  may be regarded as an isomorphism between the  $i$ -packages of  $w$  and  $E_i(w)$  that preserves  $\sigma_1, \dots, \sigma_{i-3}, \sigma_{i+2}, \dots, \sigma_{N-1}$ . If the four vertices in Figure 15 have isomorphic  $i$ -packages, we can swap all  $i$ -edges on the corresponding  $i$ -packages while maintaining axioms 2 and 5.

By axioms 2 and 5,  $E_h$  commutes with  $E_j$  whenever  $h \leq i-3$  and  $j \geq i+3$ . Bearing this in mind, the two halves of an  $i$ -package, namely  $E_2 \cup \dots \cup E_{i-3}$  and  $E_{i+3} \cup \dots \cup E_{n-1}$ , can be and often are handled separately in the following sections. Most often, establishing results for  $E_{i+3} \cup \dots \cup E_{n-1}$  is trivial, though the same results for  $E_2 \cup \dots \cup E_{i-3}$  may require considerable work.

To track axiom 3 throughout the transformation process, it is helpful to consider the following reformulation: For  $\{w, x\} \in E_i$ , at least one of  $w$  or  $x$  admits an  $i \pm 1$ -neighbor. To be more precise, if  $i > 2$ , then at least one of  $w$  or  $x$  admits an  $i-1$ -neighbor, and if  $i < N-1$ , then at least one of  $w$  or  $x$  admits an  $i+1$ -neighbor. To see the equivalence, note that by axiom 1, neither  $w$  nor  $x$  will admit an  $i-1$ -neighbor if and only if  $\sigma(w)_{i-2} = \sigma(w)_{i-1}$  and  $\sigma(x)_{i-2} = \sigma(x)_{i-1}$ . By axioms 1 and 2, this implies  $\sigma(w)_{i-2} = \sigma(w)_{i-1} = -\sigma(x)_{i-1} = -\sigma(x)_{i-2}$ . The analogous argument holds for  $i+1$ . Therefore we will often prove that axiom 3 holds by showing that at least one of  $w$  and  $E_i(w)$  admits an  $i-1$ -neighbor and at least one admits an  $i+1$ -neighbor. In addition, we often utilize the observation that both  $w$  and  $E_i(w)$  admit an  $i-1$ -neighbor if and only if  $\sigma(w)_{i-2} = -\sigma(E_i(w))_{i-2}$  and  $w$  and  $E_i(w)$  admit an  $i+1$ -neighbor if and only if  $\sigma(w)_{i+1} = -\sigma(E_i(w))_{i+1}$ .

For a signed, colored graph of type  $(n, n)$  satisfying axiom 1, axiom 3 is implied by axiom 4 and even by the weaker local Schur positivity condition. Indeed, if neither  $w$  nor  $E_i(w)$  admits an  $i-1$ -neighbor (resp.  $i+1$ -neighbor) then the connected component of  $E_{i-1} \cup E_i$  (resp.  $E_i \cup E_{i+1}$ ) containing  $w$  consists solely of  $w$  and  $E_i(w)$  forcing the restricted degree 4 generating function to be  $Q_{++-} + Q_{--+}$ , which is not Schur positive. The requirement that the graph be of type  $(n, n)$  is necessary in order to ensure that  $E_{i+1}$  edges exist in the graph. If the graph is of type  $(n, N)$  with  $n < N$ , then neither local Schur positivity nor axiom 4 is enough to ensure axiom 3.

To handle local Schur positivity, we introduce the notion of the  $i$ -type of a vertex. In the case of a dual equivalence graph, a vertex that is part of a double edge for  $E_{i-1}$  and  $E_i$  has  $i$ -type W (compare Figure 7 with  $i$ -type W in Figure 17), and otherwise the  $i$ -type of a vertex determines the shape of the connected component of  $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$  containing the vertex (compare Figure 8 with  $i$ -types A, B, and C in Figure 17). More generally, we have the following.

**Definition 5.2.** Let  $\mathcal{G}$  be a signed, colored graph of type  $(n, N)$  satisfying axioms 1, 2, 3 and 5. For  $i \leq n$  with  $i < N$ , the  $i$ -type of a vertex  $w$  of  $\mathcal{G}$  is defined to be

- $i$ -type W if  $\sigma(w)_i = -\sigma(E_{i-1}(w))_i$ ;
- $i$ -type A if  $\sigma(w)_i = \sigma(E_{i-1}(w))_i$  and  $w$  does not admit an  $i-2$ -neighbor;
- $i$ -type B if  $\sigma(w)_i = \sigma(E_{i-1}(w))_i$  and  $w$  admits an  $i-2$ -neighbor and if  $w$  admits an  $i-1$ -neighbor, then  $\sigma(w)_{i-1} = -\sigma(E_{i-2}(w))_{i-1}$ ; otherwise,  $\sigma(w)_i = -\sigma(E_{i-1}E_{i-2}(w))_i$ ;
- $i$ -type C if  $\sigma(w)_i = \sigma(E_{i-1}(w))_i$  and  $w$  admits an  $i-2$ -neighbor and if  $w$  admits an  $i-1$ -neighbor, then  $\sigma(w)_{i-1} = \sigma(E_{i-2}(w))_{i-1}$ ; otherwise,  $\sigma(w)_i = \sigma(E_{i-1}E_{i-2}(w))_i$ .

The  $i$ -type of  $w$  is determined by the connected component of  $E_{i-2} \cup E_{i-1}$  containing  $w$ . For  $i$ -type W, if  $\sigma(w)_i = -\sigma(E_{i-1}(w))_i$ , then certainly  $E_{i-1}(w) \neq w$  so  $w$  does in fact have an  $i-1$ -neighbor. For the other  $i$ -types,  $w$  may or may not have an  $i-1$ -neighbor. For  $i$ -types B and C, if  $w$  admits an  $i-2$ -neighbor but not an  $i-1$ -neighbor, then by axiom 3,  $E_{i-2}(w)$  admits an  $i-1$ -neighbor.

Figure 17 shows the  $E_{i-2}$ ,  $E_{i-1}$  and  $E_i$  edges neighboring a vertex with a given  $i$ -type. If  $E_i$  edges do not exist in the graph, then the  $i$ -edges in Figure 17 indicate which vertices admit an  $i$ -neighbor. The top rows for  $i$ -types W and B are the possibilities in a dual equivalence graph, while the lower rows give the additional possibilities in the more general setting when axiom 4 does not hold.

*Remark 5.3.* By axioms 1, 2 and 5, edges  $E_j$  with  $j < i-4$  or  $j \geq i+2$  do not change the  $i$ -type of a vertex, i.e. the  $i$ -type of  $w$  is the  $i$ -type of  $E_j(w)$ . In contrast,  $E_{i-3}$  often changes the  $i$ -type of

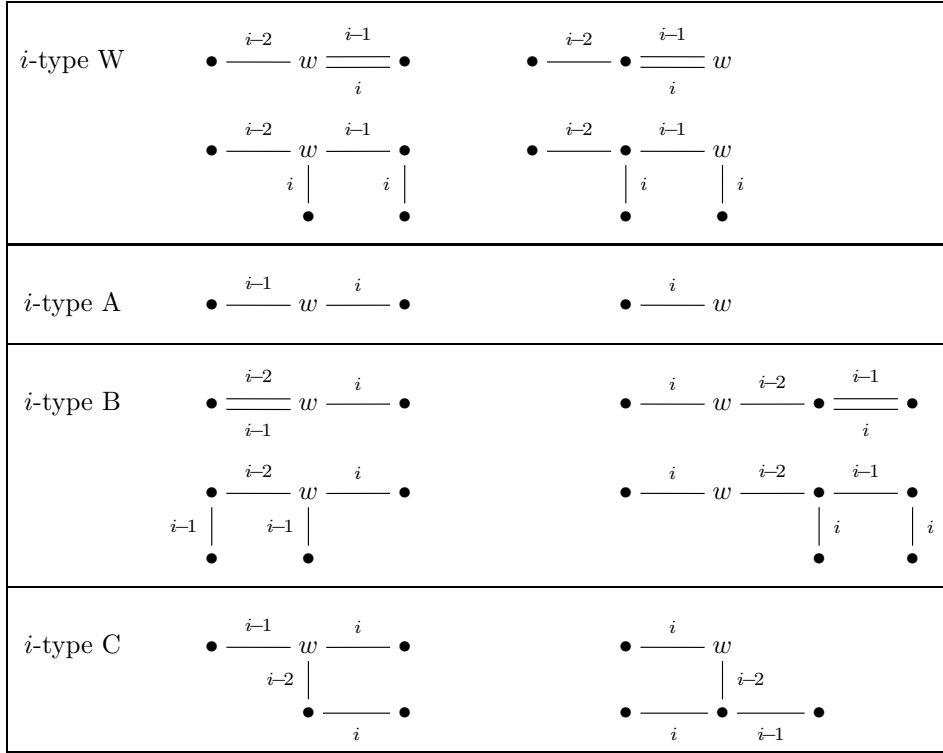


FIGURE 17. An illustration of  $i$ -type of  $w$  based on neighboring  $E_{i-2}$  and  $E_{i-1}$  edges.

a vertex as can  $E_{i+1}$ , so these cases require some care. On a more restricted level,  $w$  has  $i$ -type W if and only if  $E_{i-1}(w)$  has  $i$ -type W and both will necessarily admit an  $i$ -neighbor. Furthermore,  $w$  has  $i$ -type C if and only if  $E_{i-2}(w)$  has  $i$ -type C, and this vacuously holds for  $i$ -type A as well since, among  $i$ -types A, B and C, a vertex of  $i$ -type A is distinguished by the fact that it does not admit an  $i-2$ -neighbor. Axiom 4 is equivalent to the assertion that  $w$  and  $E_i(w)$  have the same  $i$ -type, so much of the following sections is devoted to vertices for which this is not the case.

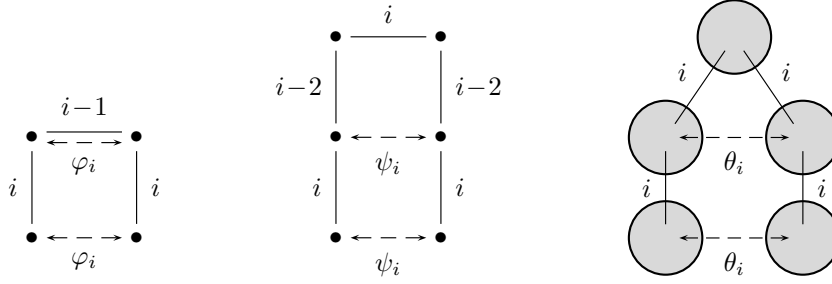
**Lemma 5.4.** *Let  $\mathcal{G}$  be a dual equivalence graph of type  $(i, N)$  with  $i < N$ . If a vertex  $w$  of  $\mathcal{G}$  has  $i$ -type W, then no vertex on the  $i$ -package of  $w$  has  $i$ -type C.*

*Proof.* By Theorem 3.9 and Lemma 3.13, we may assume  $\mathcal{G} = \mathcal{G}_{\mu, A}$  for some partition  $\mu$  of  $i$  and some augmenting tableau  $A$  containing entries  $i+1, \dots, N$ . Let  $\lambda$  be the uniquely determined shape of  $\mu$  together with the cell in  $A$  containing  $i+1$ . A tableau  $T \in \mathcal{G}_\lambda$  has  $i$ -type W if and only if both  $i-2$  and  $i+1$  lie between  $i-1$  and  $i$  in the reading word of  $T$ . From the proof of Theorem 3.14, a tableau  $T \in \mathcal{G}_\lambda$  has  $i$ -type C if and only if  $i-1$  lies between  $i$  and  $i+1$  in the reading word of  $T$ . For  $h \leq i-3$ , an  $E_h$  edge does not change the positions of entries greater than  $i-2$ , and for  $h \geq i+3$ , an  $E_h$  edge does not change the positions of entries less than  $i+2$ . In particular, the positions of  $i-1, i, i+1$  are constant on  $i$ -packages. The result now follows.  $\square$

**5.2. Three involutions to swap edges.** In this section, we present three maps,  $\varphi_i$ ,  $\psi_i$ , and  $\theta_i$ , that we use to alter the  $i$ -edges of a graph until dual equivalence graph axioms 4 and 6 hold. The basic structure of these maps is depicted in Figure 18 and described below.

Axiom 4 can be thought of as restricting the lengths of 2-color strings in the following way. Figure 7 forces the number of edges of a nontrivial connected component of  $E_{i-1} \cup E_i$  to be two,



FIGURE 18. Illustrations of the involutions  $\varphi_i$ ,  $\psi_i$ , and  $\theta_i$  used to redefine  $E_i$ .

either with three distinct vertices or forming a cycle with two vertices. The map  $\varphi_i$  swaps  $i$ -edges on connected components of  $E_{i-1} \cup E_i$  with more than two edges.

Similarly, Figure 8 forces the number of edges of a nontrivial connected component of  $E_{i-2} \cup E_i$  to be one (in the case of  $i$ -type A) or four, where there are either five distinct vertices (in the case of  $i$ -type B) or four vertices forming a cycle (in the case of  $i$ -type C). The map  $\psi_i$  swaps  $i$ -edges on connected components of  $E_{i-2} \cup E_i$  with more than four edges.

Axiom 6 can be thought of as restricting the size of  $E_2 \cup \dots \cup E_{i-1}$  isomorphism classes of a connected component of  $E_2 \cup \dots \cup E_i$  to be one. The map  $\theta_i$  swaps  $i$ -edges on connected components of  $E_2 \cup \dots \cup E_i$  with more than one member of a given  $E_2 \cup \dots \cup E_{i-1}$  isomorphism class.

We begin with the construction of the map  $\varphi_i$ , depicted in Figure 20. By dual equivalence graph axioms 1 and 2, a connected component of  $(V, \sigma, E_{i-1} \cup E_i)$  occurs in Figure 7 if and only if it does not contain a vertex  $w$  admitting an  $i-1$ -neighbor such that  $\sigma(w)_i = -\sigma(E_{i-1}(w))_i$  but  $E_{i-1}(w) \neq E_i(w)$ . Define  $W_i(\mathcal{G})$  to be the set of all such vertices bearing witness to the failure of Figure 7, i.e.

$$(5.1) \quad W_i(\mathcal{G}) = \{w \in V \mid w \text{ has } i\text{-type W but } E_{i-1}(w) \neq E_i(w)\}.$$

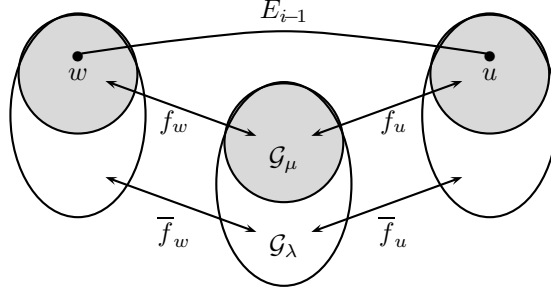
Note that  $W_i(\mathcal{G})$  is empty if and only if all connected components of  $(V, \sigma, E_{i-1} \cup E_i)$  satisfy dual equivalence graph axiom 4, or, equivalently, appear in Figure 7. Also note that  $w \in W_i(\mathcal{G})$  if and only if  $E_{i-1}(w) \in W_i(\mathcal{G})$ , and in this case both  $w$  and  $E_{i-1}(w)$  admit an  $i$ -neighbor.

**Lemma 5.5.** *Let  $\mathcal{G}$  be a signed, colored graph of type  $(n, N)$  satisfying dual equivalence axioms 1, 2 and 5, and suppose that the  $(i-2, N)$ -restriction of  $\mathcal{G}$  is a dual equivalence graph. Let  $w \in W_i(\mathcal{G})$  such that  $\sigma(v)_{i-3} = \sigma(E_{i-1}(v))_{i-3}$  for every vertex  $v$  on the  $i-1$ -package of  $w$ . Then there exists an isomorphism between the  $i$ -packages of  $w$  and  $E_{i-1}(w)$ .*

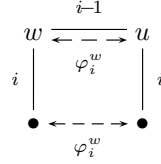
*Proof.* Fix  $w \in W_i(\mathcal{G})$  and set  $u = E_{i-1}(w)$ . Recall that  $E_{i-1}$  may be regarded as an involution on vertices that admit an  $i-1$ -neighbor. Regarded as such, by axioms 1, 2 and 5,  $E_{i-1}$  gives an involution between  $i-1$ -packages of  $w$  and  $u$ . Therefore we need only show that this isomorphism restricted to  $E_2 \cup \dots \cup E_{i-4}$  extends to an isomorphism for  $E_2 \cup \dots \cup E_{i-3}$ , since the isomorphism for  $E_{i+3} \cup \dots \cup E_{n-1}$  is already established.

By the assumption that  $\sigma(v)_{i-3} = \sigma(E_{i-1}(v))_{i-3}$  for all  $v$  on the  $i-1$ -package of  $w$ ,  $E_{i-1}$  gives an involution between the  $(i-3, i-2)$ -restrictions of the  $i$ -packages of  $w$  and  $u$ . We extend this isomorphism as illustrated in Figure 19.

By Lemma 3.13 and the hypothesis that the  $(i-2, N)$ -restriction of  $\mathcal{G}$  is a dual equivalence graph, there exist isomorphisms, say  $f_w$  and  $f_u$ , from the  $(i-3, i-2)$ -restrictions of the  $i$ -packages of  $w$  and  $u$  to the augmented dual equivalence graph  $\mathcal{G}_{\mu, A}$  for a unique partition  $\mu$  of  $i-3$  and a unique single cell augmenting tableau  $A$ . By Theorem 3.14, the two isomorphism extend consistently across  $E_{i-3}$  edges to give isomorphisms  $\bar{f}_w$  and  $\bar{f}_u$  from the  $(i-2, i-2)$ -restrictions of the connected components containing  $w$  and  $u$ , respectively, to  $\mathcal{G}_\lambda$  where  $\lambda$  is the shape of  $\mu$  augmented by  $A$ . In particular, the composition of these isomorphisms gives an isomorphism between the  $(i-2, i-2)$ -restrictions of the  $i$ -packages of  $w$  and  $u$ .  $\square$

FIGURE 19. Extending the isomorphism of  $i-1$ -packages to an isomorphism of  $i$ -packages

If every connected component of  $E_{i-2} \cup E_{i-1}$  appears in Figure 7 and  $\sigma(v)_{i-3} = -\sigma(E_{i-1}(v))_{i-3}$ , then  $E_{i-1}(v) = E_{i-2}(v)$ , so, by axiom 2,  $\sigma(v)_i = \sigma(E_{i-1}(v))_i$ . In particular, since  $\sigma_i$  is constant on  $i$ -packages, no vertex on the  $i$ -package of  $v$  has  $i$ -type W. Therefore, the hypotheses of Lemma 5.5 are satisfied whenever  $w$  has  $i$ -type W and the components of  $E_{i-2} \cup E_{i-1}$  appear in Figure 7.

FIGURE 20. An illustration of the involution  $\varphi_i^w$ , with  $w \in W_i(\mathcal{G})$  and  $u = E_{i-1}(w)$ .

For any vertex  $w \in W_i(\mathcal{G})$ , denote the isomorphism of Lemma 5.5 by  $\phi$ , and let  $u = E_{i-1}(w)$ , as depicted in Figure 20. Since the isomorphisms for  $w$  and  $u$  are inverse to one another, we abuse notation by letting  $\phi$  denote either. We use  $\phi$  to define an involution  $\varphi_i^w$  on all vertices of  $V$  admitting an  $i$ -neighbor as follows.

$$(5.2) \quad \varphi_i^w(v) = \begin{cases} \phi(v) & \text{if } v \text{ lies on the } i\text{-package of } w \text{ or } u, \\ E_i \phi E_i(v) & \text{if } E_i(v) \text{ lies on the } i\text{-package of } w \text{ or } u, \\ E_i(v) & \text{otherwise.} \end{cases}$$

Define  $E'_i$  to be the set of pairs  $\{v, \varphi_i^w(v)\}$  for each  $v$  admitting an  $i$ -neighbor. Define a signed, colored graph  $\varphi_i^w(\mathcal{G})$  of type  $(n, N)$  by

$$(5.3) \quad \varphi_i^w(\mathcal{G}) = (V, \sigma, E_2 \cup \dots \cup E_{i-1} \cup E'_i \cup E_{i+1} \cup \dots \cup E_{n-1}).$$

The following result follows from Lemma 5.5 and the definition of  $\varphi_i^w$  on  $i$ -packages.

**Proposition 5.6.** *Let  $\mathcal{G}$  be a locally Schur positive graph of type  $(n, N)$  satisfying dual equivalence axioms 1, 2 and 5, and suppose that the  $(i-2, N)$ -restriction of  $\mathcal{G}$  is a dual equivalence graph. If  $w \in W_i(\mathcal{G})$  and  $\sigma(v)_{i-3} = \sigma(E_{i-1}(v))_{i-3}$  for every vertex  $v$  on the  $i-1$ -package of  $w$ , then  $\varphi_i^w(\mathcal{G})$  also satisfies dual equivalence graph 1, 2 and 5.*

For the second transformation,  $\psi_i$ , we consider components of  $E_{i-2} \cup E_{i-1} \cup E_i$  that do not appear in Figure 8. Assuming connected components of  $E_{i-1} \cup E_i$  all appear in Figure 7, this happens precisely when two vertices with different  $i$ -types are paired by an  $i$ -edge or a pairing of  $i$ -type C forms a cycle with more than four edges. Among  $i$ -types A, B and C,  $i$ -type A vertices are distinguished by their lack of  $i-2$ -neighbors, so a mismatch can only occur between  $i$ -type B and  $i$ -type C. Therefore we focus our attention on vertices with  $i$ -type C that do not conform to Figure 8.

Define  $X_i(\mathcal{G})$  to be those vertices of  $\mathcal{G}$  that bear witness to the failure of Figure 8, i.e.

$$(5.4) \quad X_i(\mathcal{G}) = \{x \in V \mid x \text{ has } i\text{-type C and no } i-1 \text{ neighbor, and } E_{i-2}E_i(x) \neq E_iE_{i-2}(x)\}.$$

Note that if  $W_i(\mathcal{G})$  is empty, then  $X_i(\mathcal{G})$  is empty if and only if all connected components of  $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$  satisfy dual equivalence graph axiom 4, or, equivalently, appear in Figure 8.

**Lemma 5.7.** *Let  $\mathcal{G}$  be a signed, colored graph of type  $(n, N)$  satisfying dual equivalence axioms 1, 2, 3 and 5, and suppose that the  $(i-2, N)$ -restriction of  $\mathcal{G}$  is a dual equivalence graph. For  $x \in X_i(\mathcal{G})$  such that  $E_i(x)$  has  $i$ -type C and  $\sigma(v)_{i-4} = \sigma(E_{i-2}(v))_{i-4}$  for  $v = x, E_i(x)$ . Then there exists an isomorphism between the  $i$ -packages of  $E_{i-2}(x)$  and  $E_{i-2}E_i(x)$ .*

*Proof.* Fix  $x \in X_i(\mathcal{G})$  and set  $u = E_i(x)$ . Since  $x$  has  $i$ -type C,  $x$  must admit an  $i-2$ -neighbor. Since  $x$  does not admit an  $i-1$ -neighbor,  $\sigma(x)_{i-2} = \sigma(u)_{i-2}$ , and, by dual equivalence axiom 2,  $\sigma(x)_{i-3} = \sigma(u)_{i-3}$ . By axiom 1,  $u$  also admits an  $i-2$ -neighbor. Therefore both  $E_{i-2}(x)$  and  $E_{i-2}(u)$  make sense. By axioms 1, 2 and 5,  $E_i$  may be regarded as an isomorphism between the  $i$ -packages of  $x$  and  $u$ . By the same axioms, this isomorphism restricted to  $E_{i+3} \cup \dots \cup E_{n-1}$  extends across  $E_{i-2}$ . Therefore we focus our attention on the restriction to  $E_2 \cup \dots \cup E_{i-3}$ .

By Theorem 3.9, the connected components of the  $(i-2, i-2)$ -restriction of  $\mathcal{G}$  containing  $x$  and  $u$  are both isomorphic to  $\mathcal{G}_\mu$  for the same partition  $\mu$  of  $i-2$ . Denote these isomorphisms by  $f_x$  and  $f_u$ , respectively. By Lemma 5.4,  $x$  cannot lie on the  $i$ -package of a vertex with  $i$ -type W. In particular,  $\sigma(v)_{i-2} = \sigma(E_i(v))_{i-2}$  for every vertex  $v$  on the  $i$ -package of  $x$ . By Lemma 3.13, the connected components of the  $(i-2, i-1)$ -restriction of  $\mathcal{G}$  containing  $x$  and  $u$  are both isomorphic to  $\mathcal{G}_{\mu, A}$  for the same augmenting tableau  $A$  consisting of a single cell containing  $i-1$ . Let  $\lambda$  be the shape of  $\mu$  augmented by  $A$ .

Since the  $(i-1, i-1)$ -restriction of  $\mathcal{G}$  satisfies the hypotheses of Theorem 3.14, the isomorphisms  $f_x$  and  $f_u$  extend to morphisms  $\bar{f}_x$  and  $\bar{f}_u$  from the connected components of the  $(i-1, i-1)$ -restriction of  $\mathcal{G}$  containing  $x$  and  $u$  to  $\mathcal{G}_\lambda$ . The picture is very similar to Figure 19, though now the top map is  $E_i$  and the extended maps are surjective though not necessarily injective. Despite the lack of injectivity, the uniqueness of  $\lambda$  and the extended maps ensures that the  $(i-2, i-1)$ -restriction of  $\mathcal{G}_\lambda$  containing  $E_{i-2}(x)$  is isomorphic to the  $(i-2, i-1)$ -restriction of  $\mathcal{G}_\lambda$  containing  $E_{i-2}(u)$ , thereby establishing the desired isomorphism of  $i$ -packages.  $\square$

While dual equivalence axiom 3 for  $E_2 \cup \dots \cup E_{i-2}$  and the fact that  $x$  does not admit an  $i-1$ -neighbor ensures that  $E_{i-2}(x)$  admits an  $i$ -neighbor, nothing in Lemma 5.7 nor in the definition of  $X_i(\mathcal{G})$  ensures that  $E_{i-2}E_i(x)$  admits an  $i$ -neighbor. When  $E_{i-2}E_i(x)$  does not admit an  $i$ -neighbor, we must have  $\sigma(E_i(x))_{i-1} = -\sigma(E_{i-2}E_i(x))_{i-1}$ , i.e.  $E_i(x)$  has  $i-1$ -type W. We are only concerned with the case when  $E_{i-2}E_{i-1}E_{i-2}E_i(x)$  does not admit an  $i-1$ -neighbor, and so must admit an  $i$ -neighbor by axioms 1 and 2, as depicted in the right hand side of Figure 21.

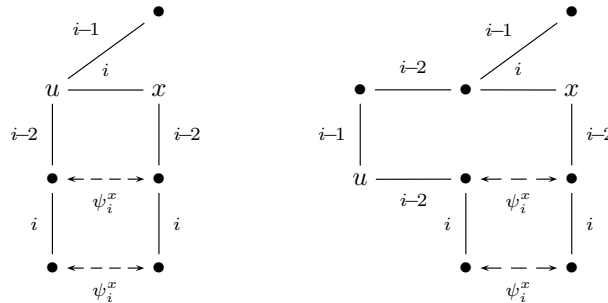


FIGURE 21. An illustration of  $\psi_i^x$  where  $x \in X_i(\mathcal{G})$  and  $u = E_i(x)$  if  $E_{i-2}E_i(x)$  admits an  $i$ -neighbor (left) or  $u = E_{i-1}E_{i-2}E_i(x)$  otherwise (right).

**Lemma 5.8.** *Let  $\mathcal{G}$  be a signed, colored graph of type  $(n, N)$  satisfying dual equivalence axioms 1, 2, 3 and 5, and suppose that the  $(i-2, N)$ -restriction of  $\mathcal{G}$  is a dual equivalence graph. Let  $x \in X_i(\mathcal{G})$*

such that  $E_i(x)$  has  $i-1$ -type  $W$  and  $\sigma(v)_{i-4} = \sigma(E_{i-2}(v))_{i-4}$  for  $v = x, E_i(x)$ . Then the  $i$ -package of  $E_{i-2}(x)$  is isomorphic to the  $i$ -package of  $E_{i-2}E_{i-1}E_{i-2}E_i(x)$ .

*Proof.* Fix  $x \in X_i(\mathcal{G})$  and set  $u = E_{i-1}E_{i-2}E_i(x)$ . We begin by showing that the  $i$ -packages of  $E_i(x)$  and  $u$  are isomorphic. Since  $E_i(x)$  has  $i-1$ -type  $W$  and  $\sigma(E_i(x))_{i-4} = \sigma(E_{i-2}E_i(x))_{i-4}$ , Lemma 5.5 applies, and so the  $i-1$ -package of  $E_i(x)$  is isomorphic to the  $i-1$ -package of  $E_{i-2}E_i(x)$ . By axioms 2 and 5,  $E_{i-1}$  gives an isomorphism between the  $i-1$ -packages of  $E_{i-2}E_i(x)$  and  $u$ , and so, by transitivity, the  $i-1$ -packages of  $E_i(x)$  and  $u$  are isomorphic. By axiom 2,  $\sigma(E_i(x))_{i-2} = -\sigma(E_{i-2}E_i(x))_{i-2} = \sigma(u)_{i-2}$ . In particular, this gives an isomorphism between the  $(i-3, i-2)$ -restrictions of the  $i$ -packages of  $E_i(x)$  and  $u$ . By the same argument used in the proof of Lemma 5.5, we invoke Lemma 3.13 and the hypothesis that the  $(i-2, N)$ -restriction of  $\mathcal{G}$  is a dual equivalence graph to extend this to an isomorphism between the  $i$ -packages of  $E_i(x)$  and  $u$ .

By axiom 3 and the assumption that  $E_i(x)$  has  $i-1$ -type  $W$ ,  $\sigma(E_i(x))_{i-1} = -\sigma(E_{i-2}E_i(x))_{i-1}$ , and by axiom 2,  $-\sigma(E_{i-2}E_i(x))_{i-1} = \sigma(u)_{i-1}$ . By axiom 2 again,  $\sigma(E_i(x))_i = \sigma(E_{i-2}E_i(x))_i$ . In particular,  $\sigma(E_{i-2}E_i(x))_{i-1} = \sigma(E_{i-2}E_i(x))_i$ , and so, by axiom 3,  $\sigma(E_{i-2}E_i(x))_i = \sigma(u)_i$ . Summarizing, we have  $\sigma(E_i(x))_{i-2, i-1, i} = \sigma(u)_{i-2, i-1, i}$ , so we may replace  $E_i(x)$  with  $u$  in the proof of Lemma 5.7 to extend this isomorphism of  $i$ -packages across the neighboring  $E_{i-2}$  edges, thereby giving the desired isomorphism between the  $i$ -packages of  $E_{i-2}(x)$  and  $E_{i-2}(u)$ .  $\square$

For  $x \in X_i(\mathcal{G})$ , if  $E_{i-2}E_i(x)$  admits an  $i$ -neighbor, then let  $\phi$  denote the isomorphism of Lemma 5.7 and  $u = E_i(x)$ ; otherwise let  $\phi$  denote the isomorphism of Lemma 5.8 and  $u = E_{i-1}E_{i-2}E_i(x)$ , as depicted in Figure 21. Abusing notation, let  $\phi$  denote both the isomorphism from the  $i$ -package of  $x$  to the  $i$ -package of  $u$  and its inverse. Use  $\phi$  to define an involution  $\psi_i^x$  on all vertices of  $V$  admitting an  $i$ -neighbor as follows.

$$(5.5) \quad \psi_i^x(v) = \begin{cases} \phi(v) & \text{if } v \text{ lies on the } i\text{-package of } E_{i-2}(x) \text{ or } E_{i-2}(u), \\ E_i\phi E_i(v) & \text{if } E_i(v) \text{ lies on the } i\text{-package of } E_{i-2}(x) \text{ or } E_{i-2}(u), \\ E_i(v) & \text{otherwise.} \end{cases}$$

Define  $E'_i$  to be the set of pairs  $\{v, \psi_i^x(v)\}$  for each  $v$  admitting an  $i$ -neighbor. Define a signed, colored graph  $\psi_i^x(\mathcal{G})$  of type  $(n, N)$  by

$$(5.6) \quad \psi_i^x(\mathcal{G}) = (V, \sigma, E_2 \cup \dots \cup E_{i-1} \cup E'_i \cup E_{i+1} \cup \dots \cup E_{n-1}).$$

The following result is a consequence of Lemmas 5.7 and 5.8 and the definition of  $\psi_i^x$  on  $i$ -packages.

**Proposition 5.9.** *Let  $\mathcal{G}$  be a signed, colored graph of type  $(n, N)$  satisfying dual equivalence axioms 1, 2, 3 and 5 such that the  $(i-2, N)$ -restriction of  $\mathcal{G}$  is a dual equivalence graph. Let  $x \in X_i(\mathcal{G})$  such that  $\sigma(v)_{i-4} = \sigma(E_{i-2}(v))_{i-4}$  for  $v = x, E_i(x)$ . Then  $\psi_i^x(\mathcal{G})$  also satisfies axioms 1, 2, and 5.*

Both  $\varphi_i^w$  and  $\psi_i^x$  also preserve dual equivalence axiom 3, though the proof of this is postponed to the following section where we investigate when these two maps preserve local Schur positivity. For the final transformation,  $\theta_i$ , the preservation of axiom 3 is integral to the construction of the map.

Let  $\mathcal{G}$  satisfy axioms 1, 2, 3 and 5 such that the  $(i, N)$ -restriction of  $\mathcal{G}$  is a dual equivalence graph and the  $(i+1, N)$ -restriction of  $\mathcal{G}$  satisfies dual equivalence axiom 4. The hypotheses on  $\mathcal{G}$  are exactly the hypotheses of Theorem 3.14. Therefore for each connected component  $\mathcal{H}$  of the  $(i+1, i+1)$ -restriction of  $\mathcal{G}$ , there exists a (surjective) morphism  $\phi$  from  $\mathcal{H}$  to  $\mathcal{G}_\lambda$  for a unique partition  $\lambda$  of  $i+1$ , and, by Corollary 3.15, the fiber over each vertex of  $\mathcal{G}_\lambda$  has the same cardinality. By Proposition 3.5 and Theorem 3.9,  $\mathcal{H}$  satisfies axiom 6 if and only if  $\phi$  is an isomorphism.

Similar to the previous transformations, define an involution  $\theta_i$  on vertices of  $\mathcal{H}$  admitting an  $i$ -neighbor as indicated in Figure 22 and use it to redefine  $i$ -edges that are in violation of axiom 6. To do this, we need to characterize when two connected components of the  $(i, i)$ -restriction of  $\mathcal{H}$  can be paired without violating dual equivalence axiom 3. Recall that partitions of size  $i$  contained in a fixed partition of size  $i+1$  are totally ordered by dominance and, by dual equivalence axiom 2,  $\sigma_{i+1}$  is constant on connected components of the  $(i, i)$ -restriction of  $\mathcal{G}$ .

**Definition 5.10.** Let  $\mathcal{H}$  be a connected signed, colored graph of type  $(i+1, N)$  satisfying dual equivalence axioms 1, 2, 3, 4, 5 such that the  $(i, N)$ -restriction of  $\mathcal{H}$  is a dual equivalence graph. Let  $\mathcal{A}$  and  $\mathcal{B}$  be two connected components of the  $(i, N)$ -restriction of  $\mathcal{H}$ , say with  $\mathcal{A} \cong \mathcal{G}_\alpha$  and  $\mathcal{B} \cong \mathcal{G}_\beta$  where  $\alpha > \beta$  in dominance order. Then  $\mathcal{A}$  and  $\mathcal{B}$  are called *i-incompatible* if  $\sigma_{i+1}(\mathcal{A}) = -1$  and  $\sigma_{i+1}(\mathcal{B}) = +1$ . Otherwise  $\mathcal{A}$  and  $\mathcal{B}$  are called *i-compatible*.

The motivation for Definition 5.10 is that  $\mathcal{A}$  and  $\mathcal{B}$  are *i-compatible* if and only if axiom 3 is satisfied for  $i$ -edges pairing a vertex in  $\mathcal{A}$  with a vertex in  $\mathcal{B}$ . Similar to the argument in the proof of Lemma 3.13, let  $w \in \mathcal{A}$  and  $v \in \mathcal{B}$  and suppose  $\{w, v\} \in E_i$ . Given the assumptions on  $\alpha$  and  $\beta$ , we have  $\sigma(w)_{i-1, i} = +-$  and  $\sigma(v)_{i-1, i} = -+$ . Therefore axiom 3 fails for this edge if and only if  $\sigma(w)_{i-1, i, i+1} = +- -$  and  $\sigma(v)_{i-1, i, i+1} = -+ +$  if and only if  $\mathcal{A}$  and  $\mathcal{B}$  are *i-incompatible*.

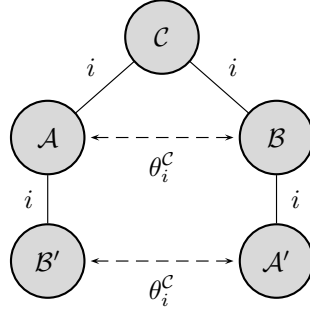


FIGURE 22. An illustration of the involution  $\theta_i^C$  where  $\mathcal{A} \cong \mathcal{A}'$  and  $\mathcal{B} \cong \mathcal{B}'$ .

Fix a connected component  $\mathcal{H}$  of the  $(i+1, i+1)$ -restriction of  $\mathcal{G}$ . If  $\sigma_{i+1}(\mathcal{B}) \equiv +1$  for every connected component  $\mathcal{B}$  of the  $(i, i)$ -restriction of  $\mathcal{H}$ , then let  $\mu$  be the maximum partition of  $i$  contained in  $\lambda$  and let  $\mathcal{C}$  be any connected component of the  $(i, i)$ -restriction of  $\mathcal{H}$  isomorphic to  $\mathcal{G}_\mu$ . Otherwise, let  $\mu \subset \lambda$  be the maximum partition (with respect to dominance order) such that there is a connected component of the  $(i, i)$ -restriction of  $\mathcal{H}$ , say  $\mathcal{C}$ , such that  $\mathcal{C} \cong \mathcal{G}_\mu$  and  $\sigma_{i+1}(\mathcal{C}) \equiv -1$ . Let  $E_i(\mathcal{C})$  be the union of all connected components  $\mathcal{B}$  of the  $(i, i)$ -restriction of  $\mathcal{H}$  such that  $\mathcal{B} \neq \mathcal{C}$  and  $\{w, u\} \in E_i$  for some  $w \in \mathcal{C}$  and some  $u \in \mathcal{B}$ . For each connected component  $\mathcal{B}'$  of the  $(i, i)$ -restriction of  $\mathcal{H}$ , let  $\phi_{\mathcal{B}'}$  be the (unique) isomorphism from  $\mathcal{B}'$  to some (unique)  $\mathcal{B} \subset E_i(\mathcal{C})$ . Finally define the involution  $\theta_i^C$  as follows.

$$(5.7) \quad \theta_i^C(u) = \begin{cases} \phi_{\mathcal{B}'}(E_i(u)) & \text{if } u \in E_i(\mathcal{C}) \text{ and } E_i(u) \in \mathcal{B}', \\ E_i(\phi_{\mathcal{B}'}(u)) & \text{if } E_i(u) \in E_i(\mathcal{C}) \text{ and } u \in \mathcal{B}', \\ E_i(u) & \text{otherwise.} \end{cases}$$

Define  $E'_i$  to be the set of pairs  $\{v, \theta_i^C(v)\}$  for all vertices  $v$  admitting an  $i$ -neighbor. Define a signed, colored graph  $\theta_i^C(\mathcal{G})$  by

$$(5.8) \quad \theta_i^C(\mathcal{G}) = (V, \sigma, E_2 \cup \dots \cup E_{i-1} \cup E'_i \cup E_{i+1} \cup \dots \cup E_{n-1}).$$

Note that  $i$ -packages are implicitly preserved for the definition of  $\theta_i$  since all  $i$ -edges on a connected component of  $E_2 \cup \dots \cup E_{i-1}$  are redefined together. Therefore the following result follows as an immediate consequence of the definition of  $\theta_i$ .

**Proposition 5.11.** Let  $\mathcal{G}$  be a signed, colored graph of type  $(n, N)$  satisfying dual equivalence axioms 1, 2, 3 and 5 such that the  $(i, N)$ -restriction is a dual equivalence graph. For  $\mathcal{H}$  a connected component of the  $(i+1, N)$ -restriction of  $\mathcal{G}$  satisfying dual equivalence graph 4 and  $\mathcal{C}$  a restricted component chosen as described, the graph  $\theta_i^C(\mathcal{G})$  also satisfies dual equivalence axioms 1, 2, 3 and 5.

**5.3. Local Schur positivity.** We have established that the maps  $\varphi_i, \psi_i$  and  $\theta_i$  preserve dual equivalence axioms 1, 2 and 5, and, in the case of  $\theta_i$ , axiom 3. In this section, we show that  $\varphi_i$  and  $\psi_i$  also maintain dual equivalence axiom 3. Though local Schur positivity may fail after one of the transformations, we show that can always remedy this by first applying a different transformation by looking carefully at when each map results in a graph that is not locally Schur positive. When investigating these situations, it is helpful to track as well the following properties of  $\mathcal{G}_{c,D}^{(k)}$ .

**Definition 5.12.** A signed, colored graph  $\mathcal{G}$  satisfies axiom 4' if the following conditions hold:

- (ax4'a) every nontrivial connected component of  $E_{i-1} \cup E_i$  has either two or four edges;
- (ax4'b) if  $\sigma(w)_{i-1} = -\sigma(E_{i-2}(w))_{i-1}$  and  $\sigma(w)_{i-2} = -\sigma(E_i(w))_{i-2}$ , then  $E_{i-1}(w) = E_{i-2}(w)$  or  $E_i(w)$ ;
- (ax4'c) if  $w$  has  $i$ -type C and  $i+1$ -type W, then  $E_{i-2}(w)$  also has  $i$ -type C and  $i+1$ -type W.

The first condition, axiom 4'a, is more a convenience than a necessity. However, if omitted, the analysis and proofs to follow become far more complicated. Note that this axiom implies local Schur positivity for degree 4 generating functions. The latter two conditions are necessary, as seen from the examples in Appendix C. Since all three conditions are local, specifically they need only be tested for connected components of  $E_{i-2} \cup E_{i-1} \cup E_i$ , they are easily verified for  $\mathcal{G}_{c,D}^{(k)}$  using either approach for establishing local Schur positivity given in Proposition 4.6. In practice, these axioms can often be shown directly similar to the verification of axiom 3.

Assume throughout that  $\mathcal{G}$  is a locally Schur positive graph satisfying dual equivalence axioms 1, 2, 3 and 5 as well as axiom 4'. We consider the maps  $\varphi_i, \psi_i$  and  $\theta_i$ , and for each map, consider  $E_{i-1} \cup E_i$  and  $E_i \cup E_{i+1}$  to establish axiom 3 and investigate degree 4 local Schur positivity, and consider  $E_{i-2} \cup E_{i-1} \cup E_i$ ,  $E_{i-1} \cup E_i \cup E_{i+1}$  and  $E_i \cup E_{i+1} \cup E_{i+2}$  to investigate degree 5 local Schur positivity. In general, we use  $\varphi_j$  to resolve problems with  $(E_{j-1} \cup E_j)$  for  $j = i, i+1$ , and we use  $\psi_j$  to resolve problems with  $(E_{j-2} \cup E_{j-1} \cup E_j)$  for  $j = i, i+1, i+2$ .

Begin with  $\varphi_i^w$  where  $w \in W_i(\mathcal{G})$  has the property that  $\sigma(v)_{i-3} = \sigma(E_{i-1}(v))_{i-3}$  for every vertex  $v$  on the  $i-1$ -package of  $w$ . For vertices  $v$  such that neither  $v$  nor  $\varphi_i(v)$  lies on the  $i$ -package of  $w$  or  $E_i(w)$ , all results follow from the hypotheses on  $\mathcal{G}$ , so, by the symmetry between  $w$  and  $E_{i-1}(w)$ , we consider only those vertices on the  $i$ -packages of  $w$  and  $E_i(w)$ . To ease notation, let  $x = E_i(w)$ ,  $u = E_{i-1}(w)$ , and  $v = E_i(u) = E_i(E_{i-1}(w))$  throughout.

First consider connected components of  $E_i \cup E_{i+1}$ . By axioms 2 and 5, both  $E_{i-1}$  and  $E_i$  commute with  $E_h$  for  $h \geq i+3$ , so if the result holds for some  $x, w, u$  and  $v$ , then it holds for any vertex on the connected component of  $E_{i+3} \cup \dots \cup E_{n-1}$  containing those vertices. Similarly, both  $E_i$  and  $E_{i+1}$  commute with  $E_h$  for  $h \leq i-3$  and  $\sigma_i$  and  $\sigma_{i+1}$  are constant on  $E_2 \cup \dots \cup E_{i-3}$ , and so if the result holds for some  $x, w, u$  and  $v$ , then it holds for any vertex on the connected component of  $E_2 \cup \dots \cup E_{i-3}$  containing those vertices. Therefore it suffices to prove the result for  $x, w, u$  and  $v$ .

Since  $w \in W_i(\mathcal{G})$ ,  $\sigma(w)_i = -\sigma(u)_i$  and, by axiom 2,  $\sigma(w)_{i+1} = \sigma(u)_{i+1}$ , so exactly one of  $w$  and  $u = \varphi_i(w)$  admits an  $i+1$ -neighbor, thereby establishing axiom 3 for  $w$  and  $u$ . Since both  $w, u \in W_i(\mathcal{G})$ , we may assume  $w$  admits an  $i+1$ -neighbor and  $u$  does not. By axiom 3 for  $\mathcal{G}$ ,  $v$  must admit an  $i+1$ -neighbor since  $u$  does not, and so at least one (though possibly both) of  $x$  and  $v = \varphi_i(x)$  admits an  $i+1$ -neighbor, thereby establishing axiom 3 for  $x$  and  $v$  as well. We now have the situation depicted in Figure 23.

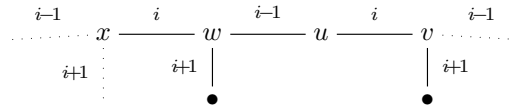


FIGURE 23. Forced  $E_{i+1}$  edges on a component of  $W_i(\mathcal{G}) \cap (E_{i-1} \cup E_i)$ , with possible edges indicated by dotted lines.

If  $E_{i+1}(w)$  admits an  $i$ -neighbor, then by axioms 1 and 2 we have  $\sigma(w)_{i-1} = -\sigma(E_{i+1}(w))_{i-1}$  and  $\sigma(w)_i = -\sigma(E_{i-1}(w))_i$ . Therefore by axiom 4'b, we must have  $x = E_i(w) = E_{i+1}(w)$ . In this case,

applying  $\varphi_i^w$  results in  $x, w, u, v$  being part of the same component of  $E_i \cup E_{i+1}$ , which is the union of two Schur positive components. In the alternative case,  $E_{i+1}(w)$  does not admit an  $i$ -neighbor, so applying  $\varphi_i^w$  results in the connected component of  $E_i \cup E_{i+1}$  containing  $w$  and  $u$  having two edges, making it a single Schur function, and so the connected component containing  $x$  and  $v$  is also locally Schur positive since the union of the two components is locally Schur positive in  $\mathcal{G}$ .

Second, consider connected components of  $E_{i-1} \cup E_i$ . By axiom 2, both  $\sigma_{i-2}$  and  $\sigma_{i-1}$  are constant on  $E_2 \cup \dots \cup E_{i-4} \cup E_{i+3} \cup \dots \cup E_{n-1}$ , and by axiom 5, those edges all commute with  $E_{i-1}$  and  $E_i$ . By dual equivalence axiom 6, we can reach any vertex on the  $i$ -package of a given vertex by crossing at most one  $i-3$ -edge, so we need only prove the result for vertices on the connected component of  $E_{i-3} \cup E_{i-1} \cup E_i$  containing  $w$ , as depicted in Figure 24. The result is immediate for  $w$  and  $u = \varphi_i^w(w)$ , since the degree 4 generating function is  $s_{(2,2)}$ , and so the result also follows for  $x$  and  $v = \varphi_i^w(x)$  since  $\mathcal{G}$  is assumed to be locally Schur positive.

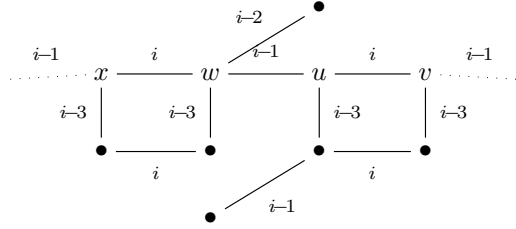


FIGURE 24. Forced  $E_{i-1}$  edges on a component of  $W_i(\mathcal{G}) \cap (E_{i-3} \cup E_{i-1} \cup E_i)$ .

Now suppose that  $w$  admits an  $i-3$ -neighbor. By Lemma 5.5, the  $i$ -packages of  $x, w, u$  and  $v$  are all isomorphic, so  $x, u$  and  $v$  also admit  $i-3$ -neighbors. By axiom 5,  $E_{i-3}(x) = E_i E_{i-3}(w)$  and  $E_{i-3}(v) = E_i E_{i-3}(u)$ , as shown in Figure 24. Since  $\sigma_i(w)_{i-3} = \sigma_i(u)_{i-3}$ , by axioms 1 and 2, exactly one of  $w$  and  $u$  admits an  $i-2$ -neighbor. Since both  $w, u \in W_i(\mathcal{G})$ , we may assume  $w$  admits an  $i-2$ -neighbor and  $u$  does not. By axiom 1, this means  $\sigma(u)_{i-3} = \sigma(u)_{i-2}$ , and so by axiom 3 for  $\mathcal{G}$ ,  $\sigma(u)_{i-2} = \sigma(E_{i-3}(u))_{i-2}$ . By axiom 2,  $\sigma(u)_{i-1} = \sigma(E_{i-3}(u))_{i-1}$ , and so  $E_{i-3}(u)$  must also admit an  $i-1$ -neighbor, thus establishing axiom 3 for  $E_{i-3}(w)$  and  $\varphi_i(E_{i-3}(w)) = E_{i-3}(u)$ .

To finish this case, we claim that if applying  $\varphi_i^w$  does not preserve local Schur positivity, then applying  $\varphi_i^{E_{i-3}(w)}$  and  $\varphi_i^{E_{i-3}(u)}$  followed by  $\varphi_i^w$  preserves local Schur positivity.

If either  $E_{i-1}(E_{i-3}(w)) = E_{i-3}(x)$  or  $E_{i-1}(E_{i-3}(u)) = E_{i-3}(v)$ , then applying  $\varphi_i^w$  results in all four of  $E_{i-3}(x), E_{i-3}(w), E_{i-3}(u)$  and  $E_{i-3}(v)$  lying on the same connected component of  $E_{i-1} \cup E_i$ . Thus assume that neither is the case. Note that  $E_{i-3}(w)$  admits an  $i-1$ -neighbor if and only if both  $w$  and  $E_{i-3}(w)$  have  $i-1$ -type C, in which case  $E_{i-3}(w)$  must also have  $i$ -type W by axiom 4'c. If  $E_{i-3}(w)$  does not admit an  $i-1$ -neighbor, then  $E_{i-3}(x)$  must, by axiom 3. If, in addition, the connected component of  $E_{i-1} \cup E_i$  containing  $E_{i-3}(u)$  has two edges, then applying  $\varphi_i^w$  again preserves local Schur positivity. In particular, if both  $E_{i-3}(w)$  and  $E_{i-3}(u)$  lie on connected components of  $E_{i-1} \cup E_i$  appearing in Figure 7, then applying  $\varphi_i^w$  preserves local Schur positivity. This proves the latter claim above: once  $\varphi_i^{E_{i-3}(w)}$  and  $\varphi_i^{E_{i-3}(u)}$  have been applied,  $\varphi_i^w$  preserves local Schur positivity.

Now suppose that at least one of  $E_{i-3}(w)$  or  $E_{i-3}(u)$  lies on a connected component of  $E_{i-1} \cup E_i$  not appearing in Figure 7. If  $E_{i-1}(E_{i-3}(u)) = E_{i-3}(w)$ , then  $E_{i-3}(w), E_{i-3}(u) \in W_i(\mathcal{G})$ , and so  $\varphi_i^w = \varphi_i^{E_{i-3}(w)}$  preserves local Schur positivity. Assume, then that  $E_{i-1}(E_{i-3}(u)) \neq E_{i-3}(w)$ . We consider in depth the two cases depicted in Figure 25, noting that if neither of these assumptions is the case, then  $\varphi_i^w$  preserves local Schur positivity.

For the left hand side, we assume that  $E_{i-3}(x)$  does not admit an  $i-1$ -neighbor, and so, by axiom 3,  $E_{i-3}(w)$  must. By earlier remarks, this implies that  $E_{i-3}(w)$  must have  $i$ -type W, and by the local Schur positivity of  $\mathcal{G}$ , the connected component of  $E_{i-1} \cup E_i$  containing  $E_{i-3}(x)$  and

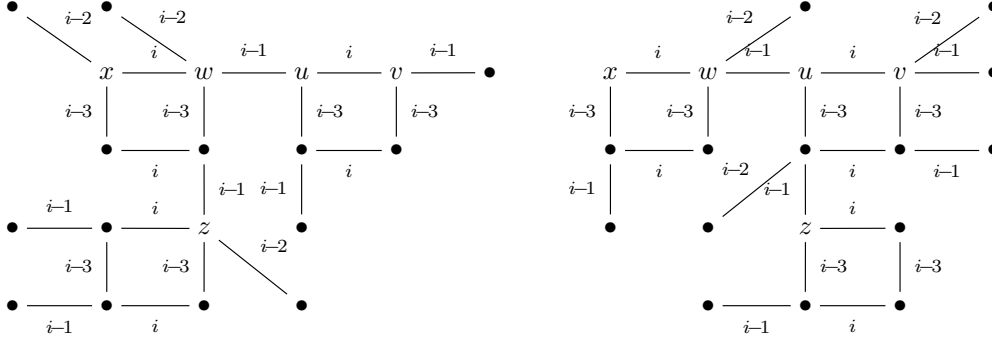


FIGURE 25. The two possible scenarios where  $\varphi_i^w$  breaks the local Schur positivity of  $E_{i-1} \cup E_i$ . The resolution is to apply  $\varphi_i^z$  first.

$E_{i-3}(w)$  must be as depicted. Since  $w$  admits an  $i-2$ -neighbor and has  $i-1$ -type C,  $E_{i-3}(w)$  does not admit an  $i-2$ -neighbor. By axiom 3 and the fact that  $E_{i-3}(x)$  does not admit an  $i-1$ -neighbor,  $\sigma(E_{i-3}(w))_{i-2} = \sigma(E_{i-3}(x))_{i-2}$ , and so, by axiom 1,  $E_{i-3}(x)$  must also not admit an  $i-2$ -neighbor. By axiom 3, this means  $x$  must admit an  $i-2$ -neighbor. Now since both  $x$  and  $w$  admit  $i-2$ -neighbors, by axiom 1, we have  $\sigma(w)_{i-2} = \sigma(x)_{i-2}$ , and so  $x$  does not admit an  $i-1$ -neighbor. Moving down the diagram, since  $E_{i-3}(w)$  does not admit an  $i-2$ -neighbor,  $z$  must admit an  $i-2$ -neighbor and an  $i-3$ -neighbor by axiom 3. By axiom 2,  $E_i(z)$  must also admit an  $i-2$ -neighbor, and by axiom 5,  $E_{i-3}E_i(z) = E_iE_{i-3}(z)$ . Since both  $z$  and  $E_i(z)$  admit an  $i-1$ -neighbor,  $E_i(z)$  cannot admit an  $i-2$ -neighbor. Therefore axiom 3 ensures that since  $E_i(z)$  admits an  $i-1$ -neighbor, so does  $E_{i-3}E_i(z)$ . Finally, if  $E_{i-3}(z)$  admits an  $i-1$ -neighbor, then both  $z$  and  $E_{i-3}(z)$  must have  $i-1$ -type C, and so by axiom 4'c,  $E_{i-3}(z)$  must have  $i$ -type W. Therefore if  $E_{i-3}(z)$  admits an  $i-1$ -neighbor, then  $E_{i-1}E_{i-3}(z)$  admits an  $i$ -neighbor. Whether this is the case or not, applying  $\varphi_i^{E_{i-3}(w)} = \varphi_i^z$  is seen to preserve local Schur positivity across the  $E_{i-3}$  edges, thereby resolving this case.

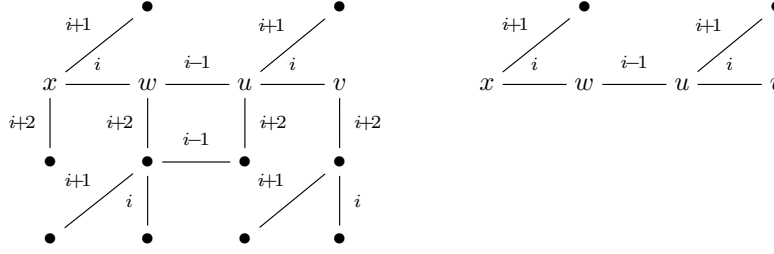
For the right hand side, we assume that  $E_{i-3}(v)$  admits an  $i-1$ -neighbor and does not have  $i$ -type W, equivalently  $E_{i-1}E_{i-3}(v)$  does not admit an  $i$ -neighbor. Therefore by the local Schur positivity of  $\mathcal{G}$ , the connected component of  $E_{i-1} \cup E_i$  containing  $E_{i-3}(x)$  and  $E_{i-3}(w)$  must be as depicted. A similar diagram chase reveals edges as depicted, where now  $z$ , and consequently  $E_i(z)$ , does not admit an  $i-2$ -neighbor, ensuring that  $E_{i-3}E_i(z)$  does not admit an  $i-1$ -neighbor. Thus applying  $\varphi_i^{E_{i-3}(u)} = \varphi_i^z$  preserves local Schur positivity across the  $E_{i-3}$  edges, thereby resolving this case.

Next, we consider the three cases necessary to establish degree 5 local Schur positivity. The key idea that persists through these three cases is that if there is an alternative path from  $x$  to  $w$  on the component in question that does not use the  $i$ -edge between  $x$  and  $w$  or if there is an alternative path from  $u$  to  $v$  on the component in question that does not use the  $i$ -edge between  $u$  and  $v$ , then the component remains connected after applying  $\varphi_i^w$ . If  $\varphi_i^w$  disconnects the components of  $E_{j-2} \cup E_{j-1} \cup E_j$  so that neither is locally Schur positive, then we first apply  $\psi_j^z$ , where  $j = i, i+1$ , or  $i+2$ , for an appropriate  $z$  so that an alternative path exists.

First consider  $E_{i-2} \cup E_{i-1} \cup E_i$ . Since exactly one of  $w$  or  $u$  admits an  $i-2$ -neighbor and  $w, u \in W_i(\mathcal{G})$ , we assume  $u$  does and  $w$  does not. There are three cases to consider, of which the first, depicted in Figure 26, is resolved with  $\psi_i^z$  where  $z = E_{i-2}(u)$  or  $E_{i-2}E_{i-1}E_{i-2}(u)$ , and the other two do not break local Schur positivity. For the first case, suppose  $\sigma(u)_{i-2} = \sigma(v)_{i-2}$ , in which case, by axioms 1 and 2,  $v$  also admits an  $i-2$ -neighbor. Either  $E_{i-2}(u)$  admits an  $i$ -neighbor and not an  $i-1$ -neighbor or, if  $u$  has  $i-1$ -type W,  $E_{i-2}E_{i-1}E_{i-2}(u)$  admits an  $i$ -neighbor and not an  $i-1$ -neighbor. Let  $z$  be  $E_{i-2}(u)$  in the former case and  $E_{i-2}E_{i-1}E_{i-2}(u)$  in the latter case. If  $E_i(z)$  has  $i$ -type B, then applying  $\varphi_i^w$  removes a component with degree 5 generating function  $s_{3,2}$  or  $s_{2,2,1}$ , in which case both components are Schur positive. In the alternative case,  $z \in X_i(\mathcal{G})$  and  $\psi_i^z$  may be applied.





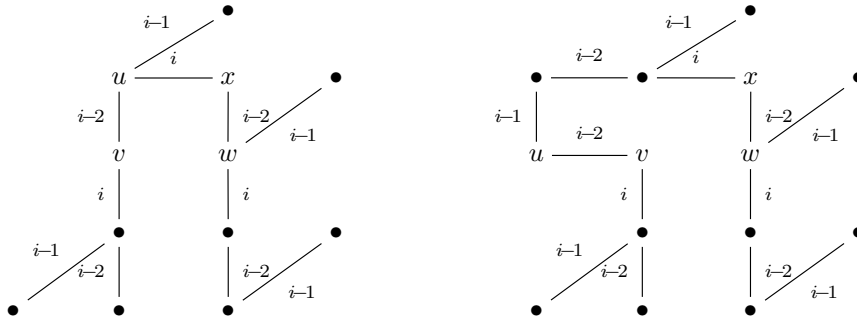
FIGURE 28. The two possibilities of  $E_{i+1} \cup E_{i+2}$  on a component of  $W_i(\mathcal{G}) \cap (E_{i-1} \cup E_i)$ .

an alternate path from  $u$  to  $v$  in  $E_i \cup E_{i+1} \cup E_{i+2}$ , so assume  $v$  admits an  $i+2$ -neighbor and not an  $i+1$ -neighbor, as depicted in Figure 28. In this case, applying  $\varphi_i^w$  cannot break the local Schur positivity if it exists in  $\mathcal{G}$ . Alternately, if none of  $x, w, u$  admits an  $i+2$ -neighbor, then, by axioms 2 and 3,  $v$  admits an  $i+1$ -neighbor if and only if  $v$  admits an  $i+2$ -neighbor. Again, if  $v$  admits an  $i+1$ -neighbor, then applying  $\varphi_{i+1}^u$  creates an alternate path from  $u$  to  $v$  in  $E_i \cup E_{i+1} \cup E_{i+2}$ , so assume that  $v$  admits neither an  $i+1$ -neighbor nor an  $i+2$ -neighbor, as depicted in Figure 28. In this case, both  $w$  and  $v$  are  $i+2$ -type A extremal points of a component of  $E_i \cup E_{i+1} \cup E_{i+2}$ , so local Schur positivity is again maintained.

All cases having been resolved, this concludes our analysis of the local Schur positivity of  $\varphi_i(\mathcal{G})$ .

Next consider  $\psi_i^x(\mathcal{G})$  for  $x \in X_i(\mathcal{G})$  and assume that the  $(i-1, N)$ -restriction of  $\mathcal{G}$  satisfies dual equivalence axiom 4. For vertices  $v$  such that neither  $v$  nor  $\varphi_i(v)$  lies on the  $i$ -package of  $E_{i-2}(x)$  or  $E_i E_{i-2}(x)$ , all results follow from the hypotheses on  $\mathcal{G}$ . To ease notation, let  $u = E_i(x)$  if  $E_{i-2}E_i(x)$  admits an  $i$ -neighbor, and otherwise let  $u = E_{i-1}E_{i-2}E_i(x)$ . Let  $w = E_{i-2}(x)$  and  $v = E_{i-2}(u)$ .

First consider components of  $E_{i-1} \cup E_i$ . Since  $x \in X_i(\mathcal{G})$ , it does not admit an  $i-1$ -neighbor. By axiom 3, both  $u$  and  $w$  must admit an  $i-1$ -neighbor. If  $E_i(w)$  admits an  $i-1$ -neighbor, then, since  $x$ , and consequently  $w$ , has  $i$ -type C,  $E_i(w)$  must have  $i$ -type  $w$  and  $E_i(w) \neq E_{i-1}(w)$ . Therefore  $w \in W_i(\mathcal{G})$  and we may first apply  $\varphi_i^w$ . Assume then that  $E_i(w)$  does not admit an  $i-1$ -neighbor. By the definition of  $\psi_i^x$ ,  $v$  admits an  $i$ -neighbor and so, by axiom 2,  $v$  does not admit an  $i-1$ -neighbor. By axiom 3,  $E_i(v)$  must admit an  $i$ -neighbor. The situation is now as depicted in Figure 29. From the figure it is clear that  $\psi_i^x$  preserves both axiom 3 and the Schur positivity of components of  $E_{i-1} \cup E_i$  containing  $E_{i-2}(x)$  and  $E_{i-2}(u)$  as well as for  $E_i E_{i-2}(x)$  and  $E_i E_{i-2}(u)$ .

FIGURE 29. Forced  $E_{i-1}$  edges on a connected component of  $X_i(\mathcal{G}) \cap (E_{i-2} \cup E_i)$ .

This result extends along  $E_2 \cup \dots \cup E_{i-4} \cup E_{i+3} \cup \dots \cup E_{n-1}$  by axioms 2 and 5. To extend the across a single  $E_{i-3}$  edge, which is all that is needed to extend the to  $i$ -packages by axiom 6, note that neither  $x$  nor  $u$  may have  $i-2$ -type W. Indeed, if so, then  $E_{i-3}(x) = E_{i-2}(x) = w$ , and, by axiom

5,  $E_{i-3}E_i(x) = E_i(w)$ . Therefore  $E_i(x)$  and  $E_i(w)$  also have  $i-2$ -type W, so  $E_{i-2}E_i(x) = E_i(w)$ , contradicting the assumption that  $x \in X_i(\mathcal{G})$ . Since neither  $x$  nor  $u$  has  $i-2$ -type W,  $\sigma(x)_{i-1} = \sigma(E_{i-3}(x))$  and  $\sigma(u)_{i-1} = \sigma(E_{i-3}(u))$ , ensuring the result across  $E_{i-3}$ .

Second, consider components of  $E_i \cup E_{i+1}$ . By axiom 2, both  $E_{i-2}$  and  $E_{i-1}$  preserve  $\sigma_{i+1}$ , so  $\sigma(x)_{i+1} = -\sigma(u)_{i+1}$  if and only if  $x$  has  $i+1$ -type W. By axiom 4'c, this implies that  $w$  also has  $i+1$ -type W, in which case both  $w$  and  $E_i(w)$  admit an  $i+1$ -neighbor thereby ensuring axiom 3 is preserved. To see the ways in which local Schur positivity may fail, we revisit Figure 25, and note that this figure is precisely an instance where  $\varphi_{i-1}$  is applicable. The resolution therefore is to apply  $\varphi_{i+1}^v$  and  $\varphi_{i+1}^w$  as needed before proceeding with  $\psi_i^x$ . This result extends along  $E_{i+3} \cup \dots \cup E_{n-1}$  by the commutativity of  $E_{i-2} \cup E_i$  ensured by axioms 2 and 5, and it extends along  $E_2 \cup \dots \cup E_{i-3}$  by the commutativity of  $E_i \cup E_{i+1}$  ensured by axioms 2 and 5.

For the three cases necessary to establish degree 5 local Schur positivity, as with  $\varphi_i$ , the key idea is to establish an alternative path from  $x$  to  $w$  or from  $u$  to  $v$ . Looking again at Figure 29, note that the local Schur positivity of components of  $E_{i-2} \cup E_{i-1} \cup E_i$  is always preserved since a single Schur positive component is being removed. The cases for  $E_{i-1} \cup E_i \cup E_{i+1}$  and  $E_i \cup E_{i+1} \cup E_{i+2}$  are straightforward and may be resolved using  $\varphi_{i+1}$ ,  $\psi_{i+1}$  or  $\psi_{i+2}$ .

Finally, we consider  $\theta_i^c(\mathcal{G})$  assuming that the  $(i+1, N)$ -restriction of  $\mathcal{G}$  satisfies dual equivalence axiom 4. We need only consider vertices in  $\mathcal{C}$ ,  $E_i(\mathcal{C})$  or  $E_i(E_i(\mathcal{C}))$ . By construction, axiom 3 is preserved and axiom 4 still holds for the  $(i+1, N)$ -restriction of  $\theta_i^c(\mathcal{G})$ , so we need only consider the local Schur positivity of  $E_i \cup E_{i+1}$  along with  $E_{i-1} \cup E_i \cup E_{i+1}$  and  $E_i \cup E_{i+1} \cup E_{i+2}$ .

If two  $E_i$  edges being interchanged have all four vertices in  $W_{i+1}(\mathcal{G})$ , then it is possible that applying  $\theta_i$  will result in one  $E_i \cup E_{i+1}$  component with three edges and another with five edges. The remedy here is as before, first apply  $\varphi_{i+1}$  to the two  $E_i$  edges and then proceed with  $\theta_i$ . The result will be a cycle of four edges for the  $E_i \cup E_{i+1}$  component, and so it will not only satisfy axiom 4'a, but axiom 4'b for  $E_i \cup E_{i+2}$  and local Schur positivity for  $E_i \cup E_{i+1} \cup E_{i+2}$  as well since two potentially distinct components have now been combined. Once again, the cases for  $E_{i-1} \cup E_i \cup E_{i+1}$  and  $E_i \cup E_{i+1} \cup E_{i+2}$  are straightforward and may be resolved using  $\varphi_{i+1}$ ,  $\psi_{i+1}$  or  $\psi_{i+2}$ .

**5.4. Transforming a D graph into a dual equivalence graph.** Finally, we are ready to show that we can apply the maps  $\varphi, \psi$  and  $\theta$  repeatedly to  $\mathcal{G}_{c,D}^{(k)}$  until dual equivalence axioms 4 and 6 hold while maintaining axioms 1, 2, 3 and 5. Thus the resulting graph is a dual equivalence graph, and so Theorem 4.7 follows from Corollary 3.10 and Proposition 4.6. In general, this method shows that any graph with the same essential properties as  $\mathcal{G}_{c,D}^{(k)}$  has a Schur positive generating function. We call such graphs *D graphs*.

**Definition 5.13.** A *D graph* is a locally Schur positive graph satisfying dual equivalence axioms 1, 2, 3 and 5 as well as axiom 4'.

Both  $\mathcal{G}_\lambda$  and  $\mathcal{G}_{c,D}^{(k)}$  are examples of D graphs. Two non-examples are given in Appendix C. The following result is the first step towards proving that a D graph can be transformed into a dual equivalence graph using the maps  $\varphi_i, \psi_i$  and  $\theta_i$ .

**Lemma 5.14.** *Let  $\mathcal{G}$  be a locally Schur positive graph of type  $(i+1, N)$  satisfying dual equivalence axioms 1, 2, 3 and 5 as well as axiom 4', and suppose that  $(i, N)$ -restriction of  $\mathcal{G}$  is a dual equivalence graph. Then for  $w \in W_i(\mathcal{G})$ ,  $W_i(\varphi_i^w(\mathcal{G}))$  is a proper subset of  $W_i(\mathcal{G})$ , and for  $x \in X_i(\mathcal{G})$ ,  $X_i(\psi_i^x(\mathcal{G}))$  is a proper subset of  $X_i(\mathcal{G})$  and  $W_i(\psi_i^x(\mathcal{G})) = W_i(\mathcal{G})$ .*

*Proof.* First consider  $\varphi_i^w$ . Since the  $(i, N)$ -restriction of  $\mathcal{G}$  is a dual equivalence graph,  $w \in W_i(\mathcal{G})$  implies that  $\sigma(v)_{i-3} = \sigma(E_{i-1}(v))_{i-3}$  for every  $v$  on the  $i-1$ -package of  $w$ . Thus  $\varphi_i^w$  can be applied to  $\mathcal{G}$  while preserving axioms 1, 2, 3 and 5. As the  $i$ -type of a vertex is determined by the connected component of  $E_{i-2} \cup E_{i-1}$  containing it, the  $i$ -type of a vertex remains unchanged by  $\varphi_i^w$ . Therefore we focus attention on the second condition for  $v \in W_i$ , namely  $E_{i-1}(v) \neq E_i(v)$ .

By axiom 5, for any  $v$  connected to  $w$  by edges in  $E_2 \cup \dots \cup E_{i-4}$ , we have  $E_{i-1}(v) \neq E_i(v)$  and  $E_{i-1}(v) = \varphi_i^w(v)$ , that is,  $v \in W_i(\mathcal{G})$  and  $v \notin W_i(\varphi_i^w(\mathcal{G}))$ . Now let  $v = E_{i-3}(w)$  and suppose that  $v$

has  $i$ -type W. We claim that  $v$  and  $w$  must have  $i-1$ -type C. Since both admit an  $i-2$ -neighbor, they cannot have  $i-1$ -type A. Since  $E_{i-3} \cup E_{i-2} \cup E_{i-1}$  satisfies axiom 4, any vertex with  $i-1$ -type W is a double edge between  $E_{i-2}$  and  $E_{i-1}$  and so, by axiom 2, cannot have  $i$ -type W. Therefore neither  $w$  nor  $v$  has  $i-1$ -type W. Since both admit an  $i-3$ -neighbor, by axiom 3 exactly one admits an  $i-2$ -neighbor, and so neither can have  $i-1$ -type B. All that remains must be the case, so both have  $i-1$ -type C. By axiom 4, this means  $E_{i-1}(v) = E_{i-3}E_{i-1}(w)$ . By axiom 5,  $E_{i-1}(w) = E_i(w)$  if and only if  $E_{i-1}(v) = E_i(v)$ , and since the former is not the case, neither is the latter. Therefore  $v \in W_i(\mathcal{G})$  and  $v \notin W_i(\varphi_i^w(\mathcal{G}))$ . Finally, if  $E_i(w)$  has  $i$ -type W, then  $E_i(w) \in W_i(\mathcal{G})$  and, by axiom 4'a,  $\varphi_i^w = \varphi_i^{E_i(w)}$ . Thus  $W_i(\varphi_i^w(\mathcal{G}))$  is indeed a proper subset of  $W_i(\mathcal{G})$ .

Second, consider  $\psi_i^x$ . If  $\sigma(x)_{i-4} = -\sigma(E_{i-2}(x))_{i-4}$ , then by axiom 4,  $E_{i-2}(x) = E_{i-3}(x)$ . Then, by axiom 5,  $E_{i-3}E_i(x) = E_iE_{i-2}(x)$ , and since both  $E_i(x)$  and  $E_iE_{i-2}(x)$  admit  $i-2$ -neighbors, by axiom 4,  $E_{i-2}E_i(x) = E_iE_{i-2}(x)$ . This argument applies equally well to  $E_i(x)$ , and so if  $x \in X_i(\mathcal{G})$ , then  $\sigma(v)_{i-4} = \sigma(E_{i-2}(v))_{i-4}$  for  $v = x, E_i(x)$  and  $\psi_i^x$  can be applied to  $\mathcal{G}$  while preserving axioms 1, 2, 3 and 5. Again, since the  $i$ -type of a vertex is determined by the connected component of  $E_{i-2} \cup E_{i-1}$  containing it, the  $i$ -type of a vertex remains unchanged by  $\psi_i^w$ . Furthermore, by the previous discussion, no  $E_{i-3}$  edge on the  $i$ -package of  $x$  or  $E_i(x)$  is part of a double edge with  $E_{i-2}$ , so whether or not the vertex admits an  $i-1$ -neighbor is preserved. Therefore we focus attention on the last condition for  $v \in X_i$ , namely  $E_{i-2}E_i(v) \neq E_iE_{i-2}(v)$ .

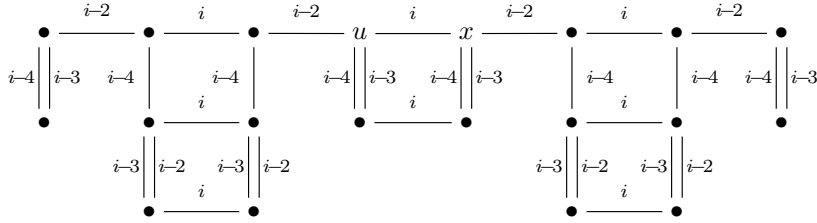


FIGURE 30. Components of  $E_{i-4} \cup E_{i-3} \cup E_{i-2}$  when  $x$  has  $i-2$ -type B.

By axiom 5, for any  $v$  connected to  $x$  by edges in  $E_2 \cup \dots \cup E_{i-5}$ , we have  $v \in X_i(\mathcal{G})$  and  $v \notin X_i(\psi_i^x(\mathcal{G}))$ . Since  $\sigma(v)_{i-4} = \sigma(E_{i-2}(v))_{i-4}$  for  $v = x, E_i(x)$ , by axiom 3 neither  $E_{i-3}(x)$  nor  $E_{i-3}E_i(x)$  admits an  $i-2$ -neighbor, so neither has  $i$ -type C. Consider the  $i-2$ -type of  $x$  and  $E_i(x)$ , which must be either B or C. By axiom 5,  $E_i$  commutes with both  $E_{i-4}$  and  $E_{i-3}$ , so  $x$  and  $E_i(x)$  have the same  $i-2$ -type. If they have  $i-2$ -type C, then the top row of Figure 30 commutes with  $E_{i-4}$ , so  $E_{i-4}(x) \in X_i(\mathcal{G})$  and  $E_{i-4}(x) \notin X_i(\psi_i^x(\mathcal{G}))$ . If they have  $i-2$ -type B, then, by axioms 4 and 5, the situation is as depicted in Figure 30 since none of the endpoints of the  $i$ -edges has  $i$ -type W by Lemma 5.4. From the figure, it is clear that applying  $\psi_i^x$  adds no vertices to  $X_i(\mathcal{G})$ , and so  $X_i(\psi_i^x(\mathcal{G}))$  is indeed a proper subset of  $X_i(\mathcal{G})$ . Moreover, by Lemma 5.4, none of the vertices involved has  $i$ -type W, and so  $W_i(\psi_i^x(\mathcal{G})) = W_i(\mathcal{G})$ .  $\square$

If the maps  $\varphi_i, \psi_i$  and  $\theta_i$  all preserved local Schur positivity, then by Lemma 5.14, for  $i$  from 2 to  $n-1$ , we could apply  $\varphi_i$  and  $\psi_i$  until  $W_i$  and  $X_i$  were both empty, then apply  $\theta_i$  until axiom 6 held, and so transform the D graph  $\mathcal{G}_{c,D}^{(k)}$  into a dual equivalence graph. Since local Schur positivity is not preserved, the transformation from D graph to dual equivalence graph is not so simple. Nonetheless, this is the basic idea behind the constructive proof of the following.

**Theorem 5.15.** *Let  $\mathcal{G} = (V, \sigma, E)$  be a D graph of type  $(n, N)$ . Then there exists a dual equivalence graph  $\tilde{\mathcal{G}} = (V, \sigma, \tilde{E})$  of type  $(n, N)$  with the same vertex set and signature function. In particular, for  $n = N$ , the sum  $\sum_{v \in V} Q_{\sigma(v)}(X)$  is symmetric and Schur positive.*

*Proof.* We proceed by constructing a sequence of signed, colored graphs  $\mathcal{G} = \mathcal{G}_2, \dots, \mathcal{G}_{n-1} = \tilde{\mathcal{G}}$  such that for each  $i = 2, \dots, n-1$ , the graph  $\mathcal{G}_i$  satisfies dual equivalence axioms 1, 2, 3 and 5 and the  $(i+1, N)$ -restriction of  $\mathcal{G}_i$  is a dual equivalence graph.

Since axioms 4 and 6 are vacuous for a graph of type  $(3, N)$ , the base case  $\mathcal{G}_2 = \mathcal{G}$  is proved. Not surprisingly, we construct  $\mathcal{G}_i$  from  $\mathcal{G}_{i-1}$  by applying the maps  $\varphi_i, \psi_i$  and  $\theta_i$ . By Propositions 5.6, 5.9 and 5.11, dual equivalence axioms 1, 2, 3 and 5 are preserved by these maps. Since each map changes only  $E_i$  edges, the  $(i, N)$ -restriction always remains a dual equivalence graph.

We focus first on the  $(i+1, N)$ -restriction. By the previous analysis, if  $W_i$  is nonempty, then for any  $w \in W_i$ ,  $\varphi_i^w$  preserves local Schur positivity of  $E_{i-1} \cup E_i$  because the situation in Figure 25 can never arise since  $E_{i-3} \cup E_{i-2} \cup E_{i-1}$  satisfies axiom 4. Furthermore, if  $\varphi_i^w$  would break the local Schur positivity of  $E_{i-2} \cup E_{i-1} \cup E_i$ , then there exists  $x \in X_i$  such that  $\psi_i^x$  preserves the local Schur positivity of  $E_{i-2} \cup E_{i-1} \cup E_i$  and afterwards  $\varphi_i^w$  does as well, as shown in Figure 26. Thus by Lemma 5.14, we may apply  $\varphi_i$ , with applications of  $\psi_i$  as needed, until  $W_i$  is empty, and then continue applying  $\psi_i$  until  $X_i$  is also empty, at which point axiom 4 holds for  $E_{i-2} \cup E_{i-1} \cup E_i$ . Then, we apply  $\theta_i^C$ , maintaining axiom 4 for  $E_{i-2} \cup E_{i-1} \cup E_i$ , until axiom 6 holds for  $E_2 \cup \dots \cup E_i$ . At this point, the  $(i+1, N)$ -restriction is a dual equivalence graph. The difficulty, then, lies with the local Schur positivity of  $E_i \cup E_{i+1}$ ,  $E_{i-1} \cup E_i \cup E_{i+1}$  and  $E_i \cup E_{i+1} \cup E_{i+2}$ .

Recall that problems with  $E_i \cup E_{i+1}$  can be resolved using  $\varphi_{i+1}$ , problems with  $E_{i-1} \cup E_i \cup E_{i+1}$  can be resolved using  $\varphi_{i+1}$  or  $\psi_{i+1}$ , and problems with  $E_i \cup E_{i+1} \cup E_{i+2}$  can be resolved using  $\varphi_{i+1}$  or  $\psi_{i+2}$ . By Propositions 5.6 and 5.9, in order to apply  $\varphi_{i+1}$  or  $\psi_{i+1}$ , the  $(i-1, N)$ -restriction must be a dual equivalence graph, and in order to apply  $\psi_{i+2}$ , the  $(i, N)$ -restriction must be a dual equivalence graph. Therefore both of these conditions are met. Moreover, the case analyses in the previous section showed that the additional signature constraints are met whenever these maps are required in order to maintain local Schur positivity. Therefore by applying these maps as described in the previous section, we can ensure the local Schur positivity of  $E_i \cup E_{i+1}$ ,  $E_{i-1} \cup E_i \cup E_{i+1}$  and  $E_i \cup E_{i+1} \cup E_{i+2}$  as well.

The final question is how to handle local Schur positivity of  $E_j \cup E_{j+1}$  and  $E_j \cup E_{j+1} \cup E_{j+2}$  for  $j \geq i+1$ . Again, from the previous analysis, we know that these cases can be resolved using  $\varphi_j$  and  $\psi_{j+1}$  for  $j \geq i+2$ , but the hypotheses needed to apply these maps are not satisfied by  $\mathcal{G}_{i-1}$ . Though the hypotheses are not met, the application of these maps is independent of edges  $E_h$  for  $h \leq i$  except in that the maps are defined on  $j$ -packages. Therefore we can apply  $\varphi_{i+2}$  and  $\psi_{i+3}$  using the  $E_i$  edges of  $\mathcal{G}_i$  but using edges  $E_h$  from  $\mathcal{G}_{i-1}$  for  $h > i$ . Using this idea of remembering where the relevant  $E_{i+1} \cup E_{i+2} \cup E_{i+3}$  edges were but using the new  $E_i$  edges that make the  $(i+1, N)$ -restriction into a dual equivalence graph allows us to apply  $\varphi_{i+2}$  and  $\psi_{i+3}$ .

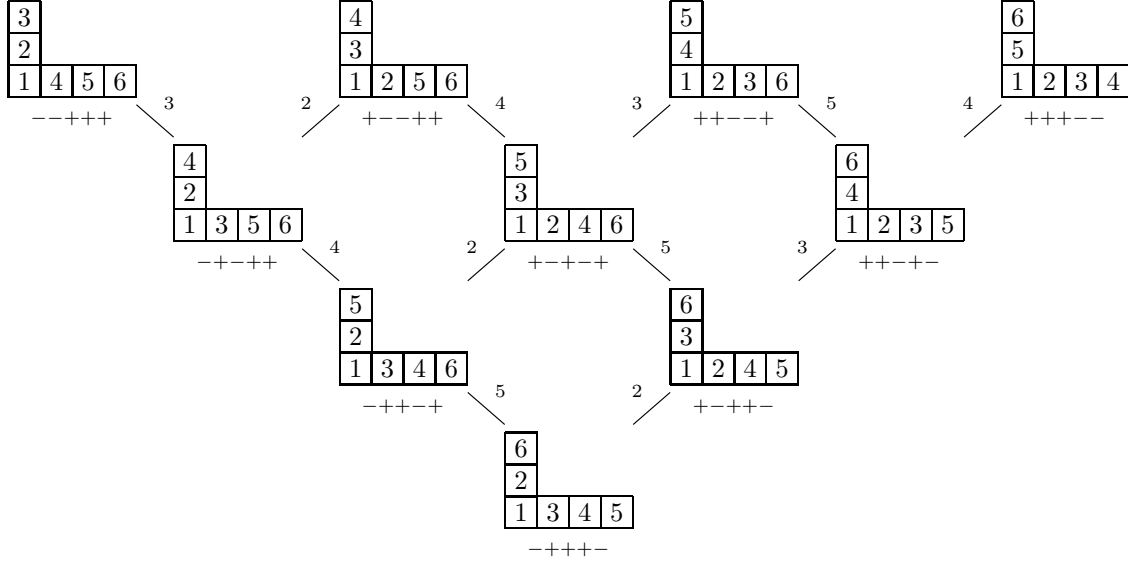
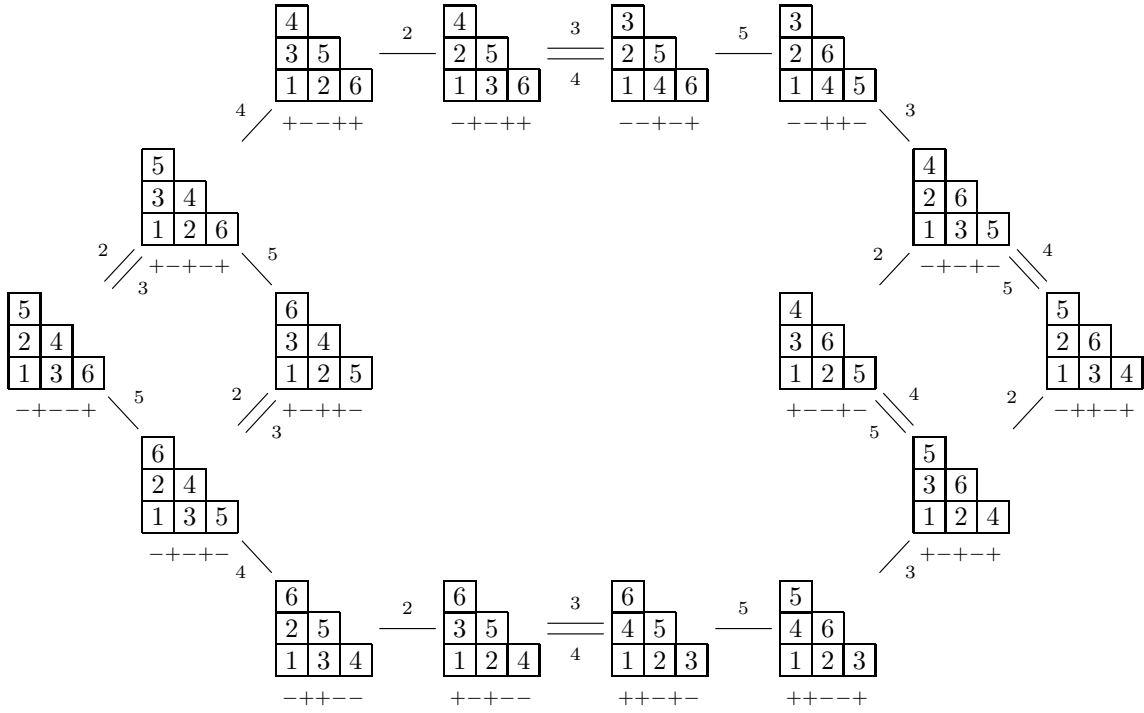
In summary, we construct  $\mathcal{G}_i$  from  $\mathcal{G}_{i-1}$  so that axioms 1, 2, 3 and 5 hold, the  $(i+1, N)$ -restriction is a dual equivalence graph, connected components of  $E_i \cup E_{i+1}$ ,  $E_{i-1} \cup E_i \cup E_{i+1}$  and  $E_i \cup E_{i+1} \cup E_{i+2}$  are locally Schur positive, and if connected components of  $E_j \cup E_{j+1}$  and  $E_j \cup E_{j+1} \cup E_{j+2}$  are not locally Schur positive for  $j \geq i+1$ , the maps  $\varphi_j$  and  $\psi_{j+1}$  for  $j \geq i+2$  can be applied to restore local Schur positivity in due course. In the end,  $\mathcal{G}_{n-1}$  is a dual equivalence graph.  $\square$

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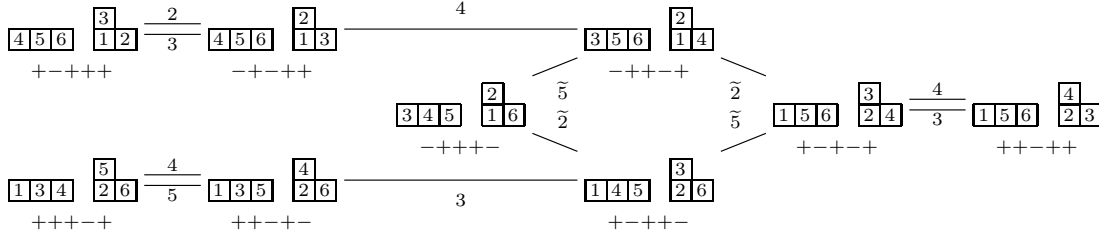
FIGURE 34. The standard dual equivalence graph  $\mathcal{G}_{3,3}$ .

FIGURE 35. The standard dual equivalence graph  $\mathcal{G}_{4,1,1}$ .FIGURE 36. The standard dual equivalence graph  $\mathcal{G}_{3,2,1}$ .

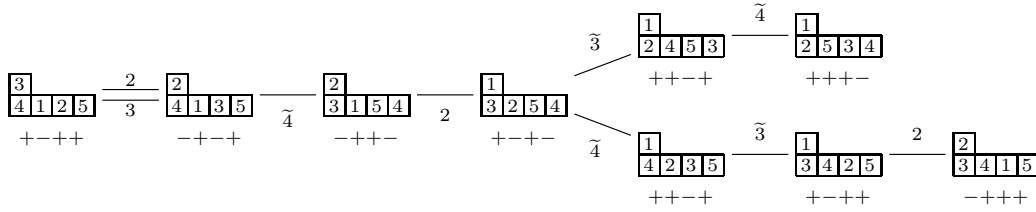


## APPENDIX B. GRAPHS FOR TUPLES OF TABLEAUX

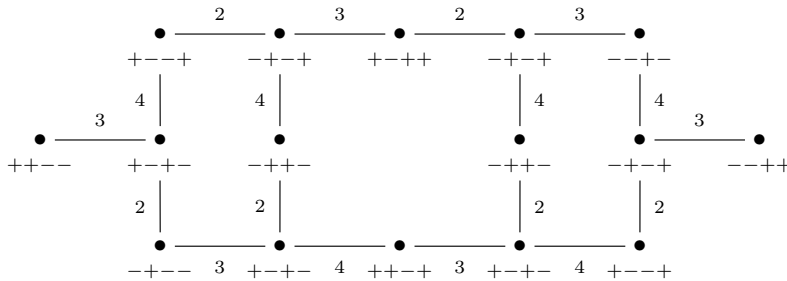
In this appendix we give several examples of connected components of the graphs  $\mathcal{G}_{c,D}^{(k)}$  constructed in Section 4 as well as the transformations to these graphs presented in Section 5. The graph in Figure 37 is a connected component of the graph on domino tableaux of shape  $((3), (2, 1))$ . Comparing this graph with the examples above, it is isomorphic to  $\mathcal{G}_{(4,2)}$ . In particular, this demonstrates Theorem 4.9 which states that the graph on domino tableaux is always a dual equivalence graph.

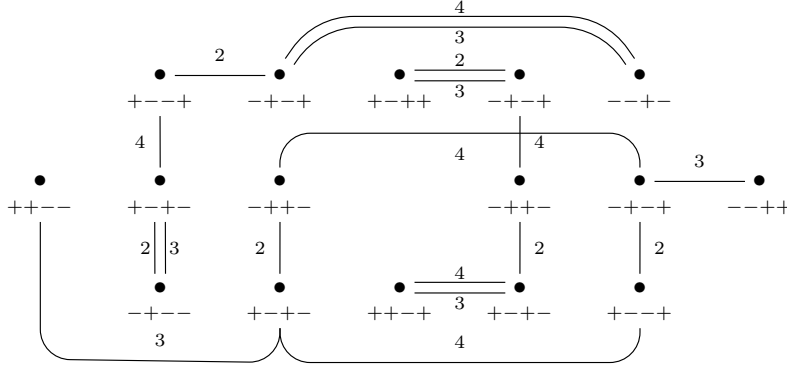
FIGURE 37. A connected component of the graph for domino tableaux of shape  $((3), (2, 1))$ .

The graph in Figure 38 is a connected component of the graph for the Macdonald polynomial  $\tilde{H}_{(4,1)}(X; q, t)$ . Note that while the generating function of the graph is  $s_{(3,2)} + s_{(4,1)}$  which indeed is Schur positive, the graph itself is not a dual equivalence graph.

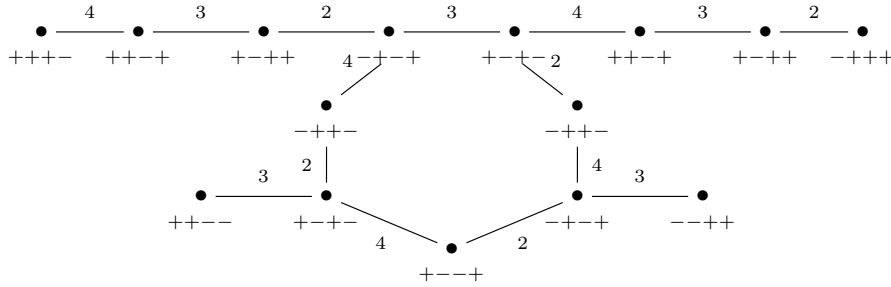
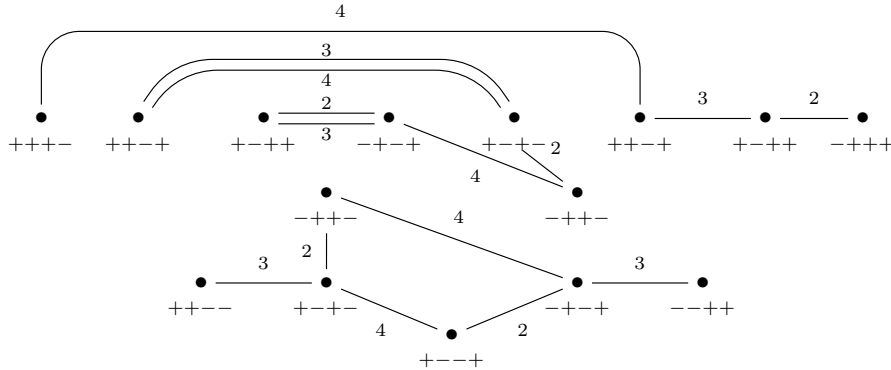
FIGURE 38. A connected component of the graph for standard fillings of shape  $(4, 1)$ .

The graph in Figure 39, which is also not a dual equivalence graph, arises as a connected component of the graph for the Macdonald polynomial  $\tilde{H}_{(5)}(X; q, t)$ . The transformation of this graph into a dual equivalence graph requires only  $\varphi_3$  and  $\varphi_4$ . The result of the transformation is the dual equivalence graph given in Figure 40. For this example, axiom 6 is immediate from axiom 4 given the size of the graph, and it is mere coincidence that  $\psi_4$  was not needed to resolve axiom 4.

FIGURE 39. A connected component for the graph for the 5-tuple  $((1), (1), (1), (1), (1))$  with generating function  $s_{3,2} + s_{3,1,1} + s_{2,2,1}$ .

FIGURE 40. The transformation of the graph in Figure 39 using  $\varphi_3$  and  $\varphi_4$ .

The graph in Figure 41 is also not a dual equivalence graph and also arises as a connected component of the graph for the Macdonald polynomial  $\tilde{H}_{(5)}(X; q, t)$ . Figure 41 shows the resulting dual equivalence graph after implementing the algorithms of Section 5, this time requiring  $\psi_4$  as well as  $\varphi_3$  and  $\varphi_4$ . Again, axiom 6 is immediate from axiom 4 given the size of the graph.

FIGURE 41. A connected component of the graph for the 5-tuple  $((1), (1), (1), (1), (1))$  with generating function  $s_{4,1} + s_{3,2} + s_{3,1,1}$ .FIGURE 42. The transformation of the graph in Figure 41 using  $\varphi_3, \varphi_4$  and  $\psi_4$ .

The final example in Figure 43, first observed by Gregg Musiker, demonstrates the necessity of axiom 6 in the definition of dual equivalence graphs. This graph arises when transforming the graph for the Macdonald polynomial  $\widehat{H}_{(6)}(X; q, t)$ . This graph satisfies dual equivalence axioms 1 through 5, but fails axiom 6. Comparing with the standard dual equivalence graph  $\mathcal{G}_{(3,2,1)}$  in Appendix A, this graph is a two-fold cover of  $\mathcal{G}_{(3,2,1)}$  as expected from its generating function  $2s_{(3,2,1)}(X)$ . Figure 44 gives the isomorphism classes of the  $(5, 6)$ -restriction of this graph.

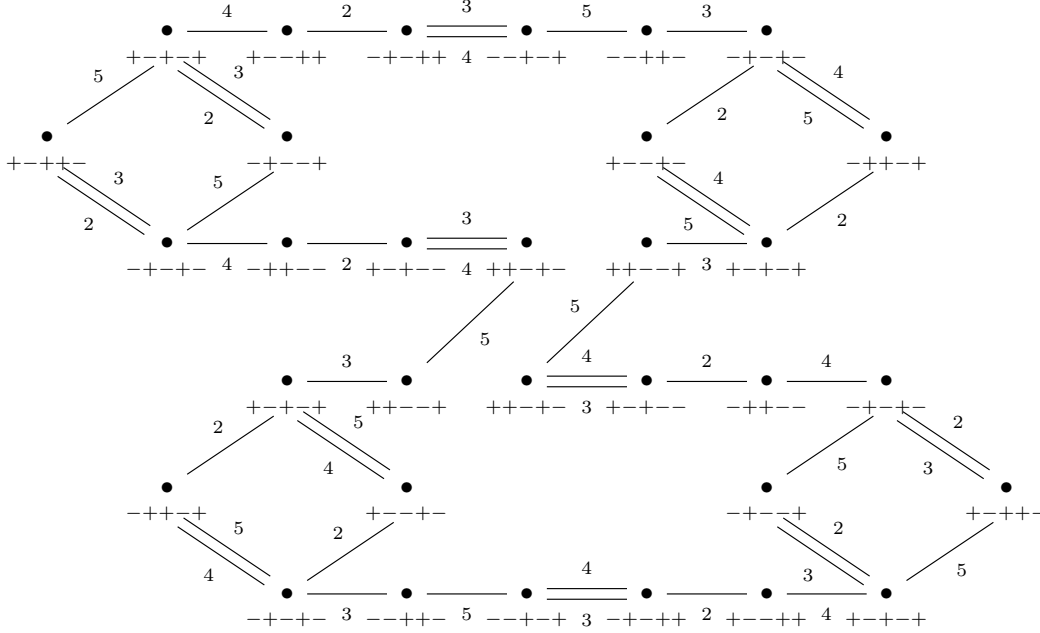


FIGURE 43. The smallest graph satisfying dual equivalence graph axioms 1 – 5 but not 6.

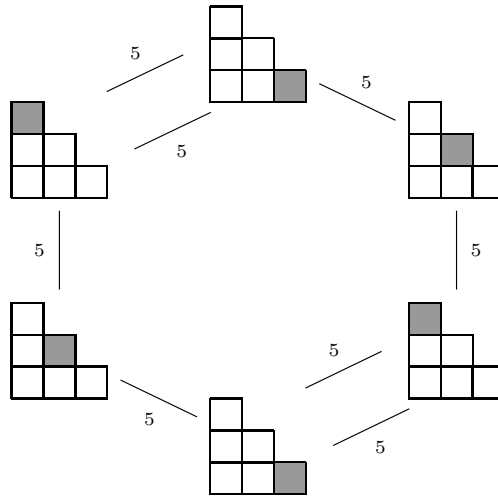


FIGURE 44. The  $(5, 6)$ -restriction of Figure 43 highlighting the two-fold cover of  $\mathcal{G}_{3,2,1}$ .

## APPENDIX C. GRAPHS FAILING AXIOM 4'

In this final appendix, we give examples of locally Schur positive graphs satisfying dual equivalence axioms 1, 2, 3 and 5 but failing axiom 4'. Not coincidentally, the transformations presented in Section 5 cannot be applied to transform these graphs into dual equivalence graphs.

Figure 45 shows a graph that violates only axiom 4'*c*. The generating function is not Schur positive. Neither  $\varphi_3$  nor  $\varphi_4$  is needed. Each of  $\varphi_5, \psi_4$  and  $\psi_5$  can be applied in exactly one place, and none of these preserves local Schur positivity. In fact, both  $\varphi_5$  and  $\psi_4$  violate axiom 3.

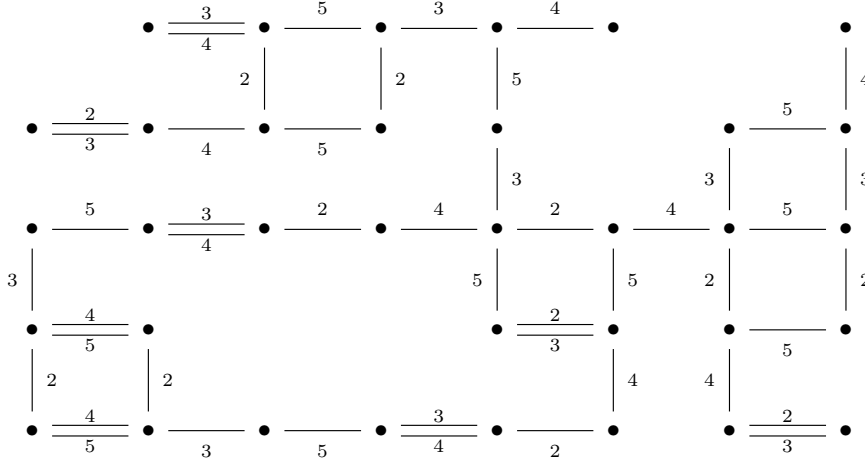


FIGURE 45. A locally Schur positive graph satisfying axioms 1, 2, 3 and 5 along with axioms 4'*a* and 4'*b* but not 4'*c*.

Figure 46 shows a graph violating only axiom 4'*b*. The generating function is not Schur positive. Here  $\varphi_4$  is needed in two places, and in both instances breaks local Schur positivity. There are two places requiring  $\varphi_5$ , however neither satisfies the hypotheses necessary to apply the map.

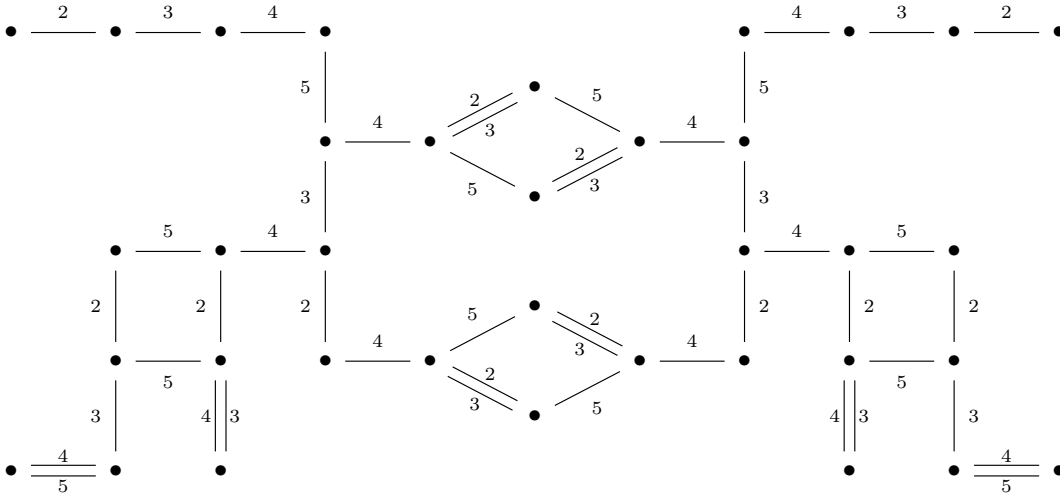


FIGURE 46. A locally Schur positive graph satisfying axioms 1, 2, 3 and 5 along with axioms 4'*a* and 4'*c* but not 4'*b*.